Matboj MatX 2021 model solutions

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OPERAČNÝ PROGRAM ĽUDSKÉ ZDROJE

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Problem 1

ABCD is a four-digit number with distinct digits A, B, C and D. When we round the number ABCD to the nearest hundred, we get number CD00. What is the largest possible value of the original number ABCD?

Solution 1

When rounding to the nearest hundred, the digit in the thousands place will change only if we were rounding up and the digit in the hundreds place was a 9. The digit B therefore has to be a 9 and after the rounding there has to be a 0 in the hundreds place, so D = 0. The digit in the thousands place had to change by 1 and we have used digits 0 and 9 so far. If we want to have as large number as possible, we need to take the two largest digits out of the remaining ones, which are 7 and 8. Since the digit in the thousands place increased after the rounding, the original number had to begin with the digit 7, so A = 7. This leaves C = 8. The original number could have been at most 7980.

Problem 2

Twenty animals are sitting around a circle and they are discussing which two of them will go for an expedition. Each animal is sitting on a rock and the rocks are labelled clockwise from 1 to 20. The selection works as follows: We start counting from the tiger who is sitting on the rock with number 1. Then, animals sitting on the every fifth rock are rejected and leave the circle. The first animals to leave will therefore be the ones on rocks with numbers 5, 10, 15, and so on. We don't count a rock if it is vacant – we count only rocks occupied with animals. The slug and the bear would really like to go on the expedition. What rocks should they sit on so that they will be the last two animals remaining?

Note: The solution should be entered as two numbers A and B separated by a comma (so in the A, B format), where A and B are the numbers of the rocks of the slug and the bear and also A < B.

Solution 2

The simplest solution in this case is perhaps not to try and analyse the whole solution, but to simulate the whole process – write down 20 numbers in a circle and cross them out in the order 5, 10, 15, 20, 6, 12, 18, 4, 13, 1, 9, 19, 11, 3, 17, 16, 2, and 8. We are left with the numbers 7 and 14.

This problem is an example of the Josephus problem which is concerned with solving such games in a more general setting.

Problem 3

Find an ordering of integers 1–12 so that the average of any two integers is never among the integers between these two numbers. For instance the ordering 1, 2, 4, 8, 3, 12, 11, 9, 10, 7, 5, 6 is incorrect because the integer 2 is somewhere in between numbers 1 and 3. You only need to find one correct ordering.

Note: The solution should be a list of 12 integers separated by commas.

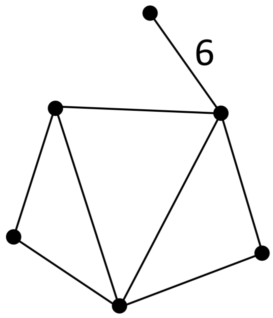
Solution 3

If we write down a random permutation of numbers from 1 to 12, we will likely end up with a pair of numbers with their average somewhere inbetween them. We have to therefore come up with a system to do this more efficiently.

Let's start by thinking about the same problem but with only three numbers. The ordering 1, 2, 3 is not good, neither is 3, 2, 1. However, 1, 3, 2 is OK. Now we will demonstrate how to transform this solution

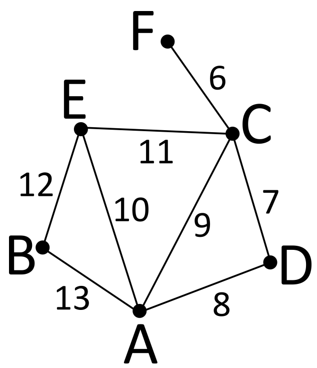
into a solution for six numbers.
The first half of the ordering of six numbers will be the ordering of the three numbers, but we will multiply each of them by two (so 2, 6, 4). Multiplying the numbers by two won't change the desired properties with regards to the averages.
The second half will be the ordering of the three numbers, but we will multiply each of them by two and subtract 1 (so 1, 5, 3). This ordering also doesn't affect the desired properties. Moreover, this step will generate odd numbers which are note present in the first half.
Therefore, we now have the ordering 2, 6, 4, 1, 5, 3. We know for sure, that the first half and the second half satisfy the desired properties. And if we choose one number in the first half (even) and one from the second half (odd), their average won't certainly be an integer.
We now have a longer ordering that still satisfies the desired properties. We now repeat the process with the ordering of 6 numbers to get an ordering of 12 numbers: 4, 12, 8, 2, 10, 6, 3, 11, 7, 1, 9, 5.
This is not the only possible ordering, we accepted any correct solution. There are 6128 possible solutions which are about 0.0013 percent of all possible orderings.

Problem 4
Andrew found a map of neighbouring villages and the roads between them. He also found a table listing lengths of the shortest routes between the villages (in kilometers). The shortest route can also go through another village. However, the map is missing the village labels... The only thing in the map is the length of one road. What is the sum of the lengths of all the roads in the map (in kilometers)?



	F	E	D	C	B
A	15	10	8	9	13
B	28	12	21	22	
C	6	11	7		
D	13	18			
E	17				

Solution 4
The correct lengths of the roads and the positions of the villages are as follows:

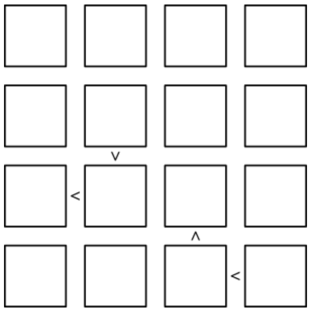


The sum of the lengths of all the roads is therefore 76 km.

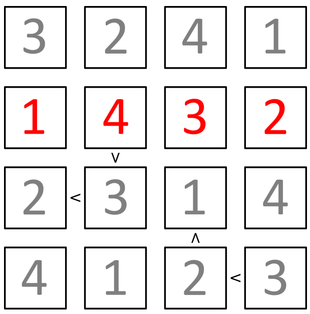
Problem 5

Solve the following puzzle. Every cell must contain an integer from 1 to 4 while each integer is used exactly once in each row and each column. Moreover, the inequalities shown between some of the cells must hold. What integers are in the second row?

Note: The solution should be entered as a four-digit number.



Solution 5



Problem 6

Augustin is observing how many tries it takes him to roll a six on a game die. If he rolls a six on the first try, he will write down number 1 on a piece of paper and he will start from the beginning. If he doesn't roll a six on the first try, he will roll again. If he rolls a six now, he will write down number 2 on the piece of paper, since this time it took him two rolls to roll a six and he will start from the beginning. If he doesn't roll a six even on the second try, he will roll for the third time – and so on. If he doesn't succeed for 30 rolls in a row, he will write down number 30 anyway and will start from the beginning. Which number will be the most frequent on Augustin's piece of paper?

Solution 6

If you roll a die many times you will find out that you will roll a six approximately once every six rolls. That means that Augustin will write down number 1 on the paper in about 1/6 of the cases (16.67 %). In the remaining 5/6 of the cases he will keep rolling. What would have to happen so that he writes down number 2 on the paper? He MUST NOT roll a six in the first roll and he HAS TO roll a six in second roll. This will happen in about $5/6 \times 1/6$ of the cases (13.8 %). Similarly, he MUST NOT roll a six in the first roll and neither in the second roll and he HAS TO roll a six in the third roll to write down number three on the paper. This will happen in about $5/6 \times 5/6 \times 1/6$ of the cases (11.58 %). We see that the larger the number, the smaller the chance – if he wants to write down a number N, the first $N - 1$ rolls must be unsuccessful (with the chance of 5/6) and only the final roll must be successful. The limit of 30 rolls is set only so that Augustin can't roll indefinitely, but in practice it doesn't change anything on the solution.

Problem 7

We have two 100-digit numbers. Each of them has exactly 5 ones among its digits with all remaining digits being zeros. We multiply these two numbers together. What will be the digit sum of this product?

Solution 7

Imagine how we would multiply these two numbers using long multiplication. We would go over the digits of the lower number. If the current digit was a zero, we wouldn't have to do anything. If it was a one, then we would set all positions where the upper number contains a one to one. Then we would sum the five numbers we got and that would give us the final answer.

However, the digit sum could break if for any of the digits the sum would be larger than 10. Fortunately the largest possible value for every digit in the final result is 5. The final digit sum will therefore be $5 \times 5 = 25$.

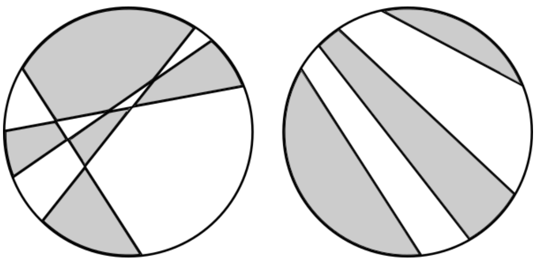
Problem 8

John and Jenny got two identical round cakes for their birthday. Jenny sliced her cake by four straight cuts so that she got the largest possible number of pieces. On the other hand, John sliced his cake by four straight cuts so that he got the smallest possible number of pieces. How many more pieces did Jenny have compared to John?

Note: All the cuts made were different, straight, vertical, complete and they all cut through the cake. Therefore don't look for any tricks in the problem.

Solution 8

Jenny could slice the cake at most in 11 pieces. We could find this number out experimentally, since the number of cuts is low. In general a well placed cut number N will add N new pieces and we started with a single piece (the whole cake). The maximum is truly $1 + 1 + 2 + 3 + 4 = 11$. If you want to know more about this topic, search for "Lazy caterer's sequence", or take a look at this Wikipedia article: https://en.wikipedia.org/wiki/Lazy_caterer%27s_sequence.
John minimized the number of pieces by making sure his cuts didn't intersect, therefore the number of John's pieces is 5. Jenny therefore had 6 more pieces than John.



Problem 9

Team “DisGustin” correctly solved 27 problems in the Matboj MatX competition. Team “Hurricane Nina” managed to correctly solve 30 problems. Team “Check It Out” correctly solved 25 problems and team “Comp Patch” correctly solved 24 problems. None of the teams used the option to skip a problem and all of the teams answered all 35 problems in the competition. Is it possible that every problem in the competition was solved incorrectly by at least one of the teams mentioned?

Solution 9

Instead of looking at the number of problems each team solved, let’s look at the numbers of problems the teams didn’t solve – this is 8, 5, 10, and 11 problems, respectively. Together, this is 34 incorrect answers. The competition has 35 problems in total, so at least one problem had to be solved correctly by all the teams. The answer is therefore NO.

Problem 10

Which cell do you have to start your journey from so that you visit all cells and finish in the cell labelled with a star? Every cell shows an arrow that tells you which cell you have to go to next.

Note: The solution should be entered as the coordinates of the starting cell, for example as A1.

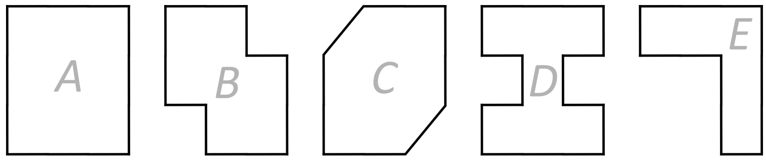
	1	2	3	4	5	6
A	↓5	→2	←2	↓3	↓1	←1
B	↓1	↓3	↓4	←2	←4	←2
C	↓2	↓1	↑1	→1	←3	←3
D	→2	→4	★	←3	↓1	↑2
E	→2	→4	↑4	↑2	←1	↑2
F	→1	↑5	→3	→1	↑2	↑5

Solution 10

The point of this problem is not in solving it (we can easily search for the previous cell of the path until we reach the beginning), but solving it efficiently. Searching for the path backwards (so called backtracking) is a very slow method in this case. It is much simpler to start in a random cell and follow the path (in average case this will give us a cell in the middle of the path) while crossing out the paths we have already visited since the path clearly doesn’t start in any of those. Once we reach the star, we again pick a random remaining cell that is not crossed-out and we follow the path until we reach the first crossed-out cell and so on. The last cell we cross out is the one we were searching for. In an average case we should be done after 4–5 iterations of this method. The path starts in the cell F4.

Problem 11
Which of the following plots of land has the smallest perimeter? And which of them has the largest perimeter?

Note: The solution should be entered as two letters separated by a comma (for example as A, B), where the first letter stands for the plot of land with the smallest perimeter and the second letter stands for the plot of land with the largest perimeter.



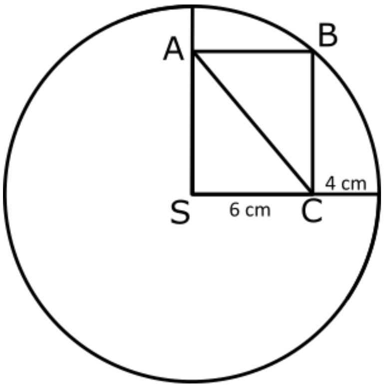
Solution 11
If we take the plot of land A and cut a square or a rectangle from its corner (= plot E) or two squares or rectangles (= plot B), the perimeter of the plot of land stays the same – we replace one edge with an edge of an equal length. However, if we cut squares or rectangles from the middle of the side (= plot D), we replace one edge with three edges. The plot of land D has therefore larger perimeter than the plot of land A. On the other hand, the plot of land C has smaller perimeter than A because of the triangle inequality (two sides of a triangle have been replaced by the third, which has to be shorter than the sum of lengths of the remaining two).

Problem 12
Mary is arranging coloured pebbles in a row. She has 4 quartz pebbles, 8 agate pebbles and 7 malachite pebbles. She wants each quartz pebble to be between an agate pebble and a malachite pebble. Also, an agate pebble and a malachite pebble must not be next to each other. In how many different ways can Mary arrange the pebbles?

Solution 12
The quartz pebbles can't be on the edges and moreover they have to be between agate and malachite pebbles. Since there is an even number of quartz pebbles, the first and the last pebbles have to be of the same type. So the only possible sequences of pebbles that are valid are:
A few agates – 1 quartz – a few malachites – 1 quartz ... – a few agates
A few malachites – 1 quartz – a few malachites – 1 quartz ... – a few malachites
If the first and the last pebbles are agates, there are together 3 groups of agates (which we have 8 of) and 2 groups of malachites (which we have 7 of), for instance AAQMMQAAAQMMMMMQAAA. It is possible to divide 8 agates into three non-empty groups in 21 ways, 7 malachites into 2 non-empty groups in 6 ways. This means $21 \times 6 = 126$ sequences of pebbles if we start with an agate pebble. With a similar solution for sequences starting with a malachite pebble, we get another 105 orderings. This means there are $126 + 105 = 231$ orderings in total.

Problem 13

A circle with a centre *S* has a rectangle *SABC* inscribed in it as shown in the picture. What is the length of the diagonal *AC* (in centimeters)?



Solution 13

If we were to calculate the length of the diagonal *AC* directly, it would take a significant amount of time and require using the Pythagorean theorem a few times. So let's look at the problem differently. The diagonal *AC* has the same length as the diagonal *SB*. The diagonal *SB* is however also the radius of the given circle. We know the radius of the circle – it is the sum of the two lengths given in the problem, hence 10 cm.

Problem 14

A sequence of numbers is made using the following rules. First, we choose the initial number – an integer larger than 0. Then we add new numbers into the sequence as follows:

1. If the last number in the sequence is even, the next number in the sequence will be its half.
2. If the last number in the sequence is odd and its first digit is even, the next number in the sequence will be the number with its digits reversed.
3. If the last number in the sequence is odd and its first digit is odd, the next number in the sequence will be the number plus 10.

We then repeat these steps. One resulting sequence could be for example 51, 61, 16, 8, 4, ... What is the smallest number we can choose at the beginning so that the sequence will never contain the number 1?

Solution 14

This is one of those problems where the solution is small enough that it can be found experimentally. We can find the solution using equations, but we assume that most teams have used the trial-and-error method. It is however important to to go through the candidate solutions efficiently, there are two rules we can use:

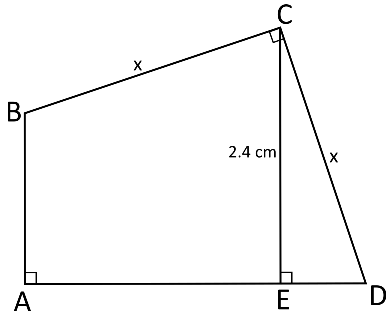
- We can skip even numbers. If there was an even number that never produced a sequence leading to 1, the same can be said about its half. This half is clearly smaller than the number itself and is therefore a better candidate as the solution of this problem.
- If we start with a number *N* and reach a number we have already eliminated as a candidate, we can eliminate also *N*.

We can quickly rule out all single-digit numbers and sequences starting with double-digit numbers quickly converge to single-digit numbers. This is not the case for the number 37, which goes to 47, which goes to 74, which goes back to 37. We have created a cycle, therefore we will never reach the number 1. The number we were looking for is therefore 37.

Problem 15

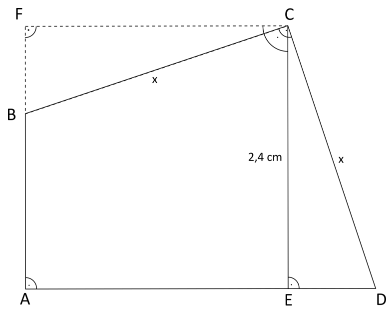
ABCD is a convex quadrilateral (this means that all of *ABCD*'s interior angles are smaller than 180°). There is a point *E* on the side *AD*, such that |*BC*| = |*CD*| and |*CE*| = 2.4 cm. All of *DCB*, *DAB*, and *CEA* are right angles. What is the area of *ABCD* in cm^2?

Note: If necessary, round the result to two decimal places.



Solution 15

The first picture shows the given problem.
If we draw an additional quadrilateral *CEAB* and rectangle *CEAF*, we see that the triangles *CED* and *CFB* are congruent (the same size, just different rotation). Why is that the case?
Firstly, the lengths of *BC* and *CD* are equal. Secondly, the angles *BCF* and *ECD* have the same size, since the angles *BCD* and *ECF* are right angles. And since the angles *BFC* and *CED* are right angles, then also the sizes of angles *FBC* and *CDE* are equal.
The triangles *CED* and *CFB* are congruent by the angle-angle-side postulate. This means they also have the same area. It is also the case that |*CE*| = |*CF*|. This means that *CEAF* is actually a square.
Now we just need to find out the area of the square *CEAF*. This is easy, since we know that its side is 2.4 cm. Its area is therefore 5.76 cm^2.



Problem 16

Usain and Asafa made it into the Olympic finals in the 100-metre sprint where 8 runners will compete. How many different possible outcomes of the race are there such that Usain will beat Asafa?

Note: We assume that no two runners had the same time.

Solution 16

How many different orderings are there in total? We can choose from 8 runners in the first place, from 7 in the second place, and so on. There is therefore $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$ orderings in total. The trick is to notice that for every ordering where Asafa beats Usain there is exactly one other ordering where Usain beats Asafa – we simply swap their places in the ordering. Therefore the number of of orderings where Usain has beaten Asafa is exactly one half of the total number of orderings, that is 20160.

Note: We also accepted $(8 \times 7)/2 = 28$ as a solution, which doesn't take the ordering of the other runners into account.

Problem 17

Stella wrote a three-digit number on a piece of paper. Then she wrote the same number again after the first number and therefore got a six-digit number. What is the least number of divisors (including 1 and the number itself) that the resulting six-digit number could have?

Solution 17

Stella got a number which we can write as $ABCABC$. However, we can also write this number as $ABC000 + ABC$, or $1000 \times ABC + ABC = 1001 \times ABC$. Stella's number will have the smallest possible number of divisors when ABC is a prime. 1001 can be factored into $7 \times 11 \times 13$. Prime numbers in the factorization of the number $ABCABC$ are therefore 7, 11, 13, and ABC. Out of these prime numbers we can create exactly $2^4 = 16$ different divisors of Stella's number.

Problem 18

100 wizards came to a wizard convention. Every wizard is either happy or sad. We know the following:

1. At least one of the wizards present is sad.
2. At least one wizard is happy in any pair of wizards.

How many wizards at the wizard convention are happy?

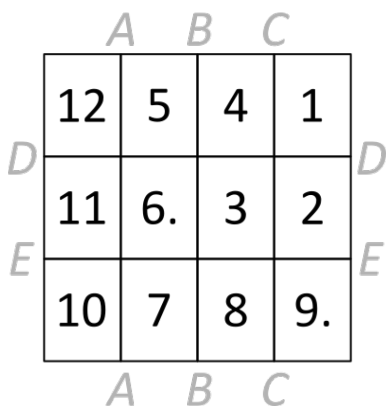
Solution 18

The sentence "at least one wizard is happy in any pair of wizards" is the same as "there is no pair with two sad wizards". If there were at least two sad wizards, there would surely be at least one pair with both wizards sad – which is prohibited by the rules. There are therefore less than 2 sad wizards. Since we know there is at least one sad wizard, it has to be exactly one wizard. This means that there are 99 happy wizards.

Problem 19

Ian bought a paper map. He unfolded it, but he doesn't remember, in what order to fold it back again. In what order does he have to fold along the folds A, B, C, D, and E so that the folded map has its parts numbered from 1 to 12 top to bottom? The number is valid for both sides of the map.

Note: The solution should be entered as a sequence of folds (for example EBADC), the folding direction doesn't have to be included.



Solution 19

The first fold must be along D, so that we get the part 2 on top of the part 1. The next fold must be along C, so that we get the part 3 on top of the part 2. The part 4 is already in the correct place, so the next fold (along B) has to get the part 5 on top of the part 4. The part 6 is already in the correct place and we get the part 7 on the part 6 by folding along E. This fold will also places the parts 8 and 9 in their correct places. The final fold along A places the parts 10, 11, and 12 in their correct places.

Problem 20

The product of twenty-five integers is 1. What is their smallest possible sum?

Solution 20

If there was a zero among the numbers, their product would also be a zero, which would contradict what we were given. If any of the numbers were (in absolute value) at least two, then also the product would be (in absolute value) at least 2. Therefore the numbers are only the numbers 1 and −1. Since we would like to get the smallest possible value of their sum, we would like to have as many −1s among them as possible. This is the case when we have 24 −1s and one positive 1. The sum of these numbers is −23.

Problem 21

Every number has its "footprint". The footprint of a number can be calculated by summing its first digit multiplied by one, the second digit multiplied by two, the third digit multiplied by three and so on until we run out of digits. For example, the footprint of 23507 is $2 \times 1 + 3 \times 2 + 5 \times 3 + 0 \times 4 + 7 \times 5 = 58$. What is the smallest positive integer that has its footprint equal to 100?

Solution 21

Let's find out how many digits the number we are looking for has. If it had just a single digits, its largest possible footprint would be $9 \times 1 = 9$. This is clearly not enough. If it had two digits, its largest possible footprint would be $9 \times 1 + 9 \times 2 = 27$. Still not enough. For 3 digits: $9 \times 1 + 9 \times 2 + 9 \times 3 = 54$.

Still not enough. For 4 digits: $9 \times 1 + 9 \times 2 + 9 \times 3 + 9 \times 4 = 90$. Still not enough. For 5 digits: $9 \times 1 + 9 \times 2 + 9 \times 3 + 9 \times 4 + 9 \times 5 = 135$. Five-digit number should therefore be enough. Now we need the footprint to be exactly 100.

To make the number as small as possible, we need to decrease the values of the digits from left to right. We can decrease the first digits at most to 1, the footprint then becomes $1\times1+9\times2+9\times3+9\times4+9\times5 = 127$. We can decrease the second digit to 0. The footprint then becomes: $1\times1+0\times2+9\times3+9\times4+9\times5 = 109$. We can then decrease the third digit by 3, so the final number is 10699.

Problem 22
A dad would like to make his quintuplets happy by buying one muffin for each of them for their birthday. But the kids like to argue. They won't argue only if all of them either get the same kind of muffin or if each of them gets a different kind of muffin. The dad went to a pastry shop which sells at least five different kinds of muffins. All of them looked delicious, so he told the shop assistant to give him X muffins chosen *randomly*. What is the lowest possible value of X so that the dad doesn't cause an argument?

Note: All remaining muffins will be eaten by mom and dad, because they also deserve something sweet.

Solution 22
The goal is for the quintuplets to either all have the same kind of muffin or all of the to have a different kind each. Let's label the muffin kinds as A, B, C, D, and E. 16 randomly chosen muffins are clearly not enough, since the shop assistant could have chosen 4 As, 4 Bs, 4 Cs, and 4 Ds. We can't divide this selection of muffins among the quintuples so that they are happy. However, the seventeenth muffin will either be the fifth of one of the kinds we already have (and the quintuples will each get one of the same kind) or it will be of the fifth kind E (and the quintuples will each get one of a different kind). The parents are going to be the winners though since they will be left with 17 – 5 = 12 muffins.

Problem 23
Joseph was given two options on how his salary could be paid to him. Option A is that he will get 4000 pounds in the first year and then every year his salary will increase by 800 pounds. Option B is that he will get 2000 pounds for the first half year and then every 6 months his salary will increase by 200 pounds. Which of the options is better for Joseph assuming he will work for at least one year?

Solution 23
Let's find out how much would Joseph make in both cases. We will group the incomes in the individual years in brackets. For the option A he would make (4000)+(4800)+(5600)+(6400)+... For the option B he would make (2000+2200)+(2400+2600)+(2800+3000)+(3200+3400)+... = 4200+5000+5800+6600+...

We see that if he worked for less than half a year, then the options A and B would be the same. However, if he worked for longer than that, the option B would be better for him.

Problem 24

A number N can be written as a sum of seven consecutive positive integers but also as a sum of eight consecutive positive integers. What is the smallest possible value of N?

Note: We don't consider 0 to be positive.

Solution 24

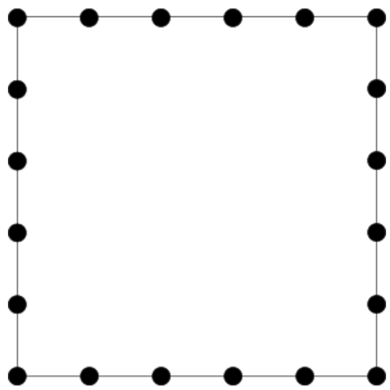
The sum of seven consecutive integers can be written as $k+(k+1)+(k+2)+(k+3)+(k+4)+(k+5)+(k+6) = 7 \times k + 21 = 7 \times (k+3)$. The number we are looking for is therefore divisible by seven.

The sum of eight consecutive integers can be written as $n + (n+1) + (n+2) + (n+3) + (n+4) + (n+5) + (n+6) + (n+7) = 8 \times n + 28 = 4 \times (2n+7)$.

The number we are looking for is therefore a multiple of 28 (but not 56, since in the second equation we could factor out only 4, not 8). The number 28 can't be written as a sum of eight consecutive integers, 56 is not an option, but the next one – 84 – satisfies both conditions (finding the summands is left as an exercise for the reader).

Problem 25

There is a square with a side 5 cm. There are 6 points on each of the sides with regular spacing of 1 cm between them. How many different segments that start in one of the points and end in one of the other points are there such that their length is 5 cm?



Solution 25

There are 6 horizontal segments. There are also 6 vertical segments. We must not forget also about 8 diagonal segments, where two always start on one side and end on the neighbouring side. Because of Pythagorean theorem, all of these are of length 5 cm, since $3^2+4^2 = 5^2$. There are therefore 6+6+8 = 20 segments in total.

Problem 26
We divided positive integers smaller than 1000 into three groups:

1. Even numbers with only even digits.
2. Odd numbers with only odd digits.
3. All remaining numbers.

What is the absolute value of the difference in sizes of the first group and the second group?

Note: The digit 0 is even.

Solution 26
For single-digit numbers, we are done quickly – we have 5 odd a 4 even numbers.
Odd two-digit numbers with odd digits only can have 5 possible digits in the tens place (1, 3, 5, 7, 9) and same for the ones place, so there is 25 of them. Even two-digit numbers with even digits only can have 4 digits in the tens place (2, 4, 6, 8) and 5 in the ones place (the same plus zero). This is 20 numbers in total.
Similarly, odd three-digit numbers with odd digits only are $5 \times 5 \times 5 = 125$ and even with even digits only are $4 \times 5 \times 5 = 100$. The difference between the sizes of the first and the seconds groups is therefore $125 + 25 + 5 - 100 - 20 - 4 = 31$.

Problem 27
Peter would like to exchange a 100-euro banknote which he got from an ATM. He would like to exchange it for smaller banknotes. There are 50, 20, 10 and 5-euro banknotes available. In how many different ways can he exchange the 100-euro banknote?

Solution 27
We can use the fact that 5 divides 10, 20, 50, and 100. This means that whatever banknotes we use, the remainder to 100 will always be payable using 5-euro banknotes (we don't even have to count them, it is sufficient to say "we will use as many as needed").
The only remaining thing is to come up with a system of enumerating the different possibilities. A good system is to write down the possibilities so that the number of the largest banknotes is decreasing (2×50, $50 + 2 \times 20 + 10$, $50 + 2 \times 20$, $50 + 20 + 3 \times 10$, $50 + 20 + 2 \times 10$, ...).
Don't forget that that number of 5-euro banknotes is given by the remainder to 100 and we can ignore it because it is fully given by the already-used banknotes. Using this system we can find out that the number of solutions is 49.

Problem 28
At the beginning there was a positive integer. We took its digits, squared (i.e. multiplied with itself) each of them, summed the intermediate results and added 1. Surprisingly we ended up with the original number. Find the largest positive integer that satisfies this.

Example: The number 24 is not a solution, since $2 \times 2 + 4 \times 4 + 1 = 21 \neq 24$.

Solution 28
The number surely didn't have four digits, since the sum of the squares of its digits is at most $4 \times 9 \times 9 + 1 = 325$. For larger numbers this problem gets even worse.
Could the number have had three digits? The largest possible value is for the number 999, which gives

$3 \times 9 \times 9 + 1 = 244$. Although this is too small, it is already a three-digit number – but with 2 in the hundreds place. Let's make a generous upper bound estimate and say that the number is at most 299. If we calculate its value, we will get $2\times2+2\times9\times9+1 = 167$. This is still less than the original number. This gave us another estimate for the number – it is at most 167. Now we see that we could get the largest possible value using the number 159, however, even that one gives a value of $1\times1+5\times5+9\times9 = 108$. This again lower the upper bound for the number we are looking for to 108. We now see that none of the three-digit numbers will work. The purpose of this paragraph was to show by decreasing the upper bound that even three-digits numbers won't work.
For two-digit numbers the organisers didn't find a simple enough solution other than trying all possible numbers from 100 downwards. The largest such number is 75.

Problem 29
We have an infinite square grid. Initially, there is one square with a number 1 in it, all remaining squares have a zero in them. Every second, a number in each square is replaced by the sum of the numbers in the four squares adjacent to it. What will be the sum of all numbers in the square grid after 10 seconds?

Solution 29
Let't focus on one cell at any given time. What will happen one second later? The value of this cell will be added to the values of the four cells around it. Moreover, the value of this call disappears, since the value of the cell itself doesn't affect its value one second later (the value of this cell will be the sum of the values in the four cells around it). This happens for every cell. This implies only one thing – every second the sum of all cells in the square grid multiplies by four. The initial sum is 1, after 10 seconds their sum will be $4^{10} = 1048576$.

Problem 30
We have a number that doesn't contain the digit 0. We add the number itself written backwards to this number. The result gives us a three-digit number which contains only the digits 6 and 9 (the number doesn't necessarily have to contain both of these digits). How many different numbers could we have had initially?

Solution 30
The original number must have three-digits, since are able to form at most the number $99 + 99 = 198$ while the smallest three-digit number containing only the digits 6 and 9 is 666. The first and the third digit of the result have to be equal – we are summing the same digits from the original number and it is not possible we went over 10 when summing them because this would add at most 1, while the digits 6 and 9 are more than 1 apart. This means that the result could be only the numbers 666, 969, 696, and 999.
If the second digit in the result is a 6, then we could not have gone over 10 when summing the digits. For the middle digit we need to su, the middle digit with itself and we get an even number. Going over 10 would give us an extra 1, which we don't want. This implies that the middle digit of the original number is a 3 and the sum of the first and the last digit is smaller than 10 (6 in the case of 666 and 9 in the case of 969, the sum couldn't have been 16 or 19). This gives us the following options: For 666 the original number was 135, 234, 333, 432, or 531. For 969 the original number was 128, 237, 336, 435, 534, 633, 732, or 831. 13 possibilities altogether.
If the second digit in the result is a 9, then it means that we had to go over 10, since 9 is an odd number. This means that the sum of the first and the third digit had to be either 16 or 19. This would mean,

however, that the sum would have four digits, which we don't want. We therefore don't add any more possibilities in this case.

Problem 31

Michael has a wooden cube with a side 30 cm which he would like to cut using straight saw cuts into 27 smaller cubes with a side of 10 cm. He can rearrange the intermediate pieces of wood between the individual cuts. What is the smallest number of cuts Michael has to make to cut the cube?

Solution 31

The fact that Michael can rearrange the pieces between the cuts is nice, but useless really. We can show this by focusing on the little central cube. Each cut will cut away at most one other neighbouring little cube from the little central cube (otherwise the cut count not be straight). This means that we need to do at least 6 cuts, since the little central cube has 6 neighbours. And since we have to do 6 cuts, we can cut the big cube easily without rearranging the pieces between the cuts.

Problem 32

How many three-digit numbers are there such that their digit in the hundreds' place is bigger than the digit in the ones' place and the digit in the tens' place is bigger than the digit in the hundreds' place?

Solution 32

The digit in the hundreds place isn't the smallest digit, therefore it can't be set to 0 and we don't have to be concerned with this corner case.

If we pick three different digits from numbers 0 to 9, we can form a single number that satisfies the given conditions.

This means that the number of such numbers is the number of ways we can pick 3 different digits from the ten possible ones. The first digit can be chosen from ten options, the second from nine, and the third only from eight. Moreover, we mustn't forget that choosing A, B, C is the same as choosing A, C, B, or B, A, C, or B, C, A, or C, A, B, or C, B, A. We have to therefore divide the total number $10 \times 9 \times 8$ by six, which gives us 120 options.

Problem 33

Laura usually returns home from work by train and Dan picks her up at 5pm. He then drives her home. One day Laura finished work sooner and she managed to catch an earlier train that arrives at her station one hour earlier. But Laura's phone battery died and she couldn't text Dan about this. The weather was nice that day so Laura decided to walk home and meet Dan on the way. When she met Dan, she got in the car and they drove home. They got home 10 minutes earlier than they usually do. How many minutes did Laura walk for?

Solution 33

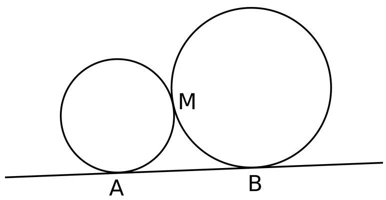
Let's look at the situation from Dan's point of view instead of Laura's. Dan will get home 10 minutes earlier than normally. This means that he saved 10 minutes on the part of the journey that goes from the point where he picked Laura to the train station and back to that point. He therefore saved 5 minutes in one direction and he picked Laura 5 minutes earlier than normally. Laura therefore had to walk for 60 minutes (she came earlier by that amount) – 5 minutes (time she saved), that is 55 minutes in total. Note that we didn't need to know the speeds of Laura and Dan and that we even don't need to assume both travelled with constant speed.

Problem 34
How many positive integers k satisfy the condition that the sum $1! + 2! + 3! + ... + k!$ is a square of a positive integer?

Note: Factorial of a positive integer n (which we write as $n!$) is calculated as a product of all positive integers from 1 to n, that is $1 \times 2 \times 3 \times ... \times n$.

Solution 34
Let's write down the first few values and focus on the final digit of the number we get. For $n = 1$ we get $1! = 1$ (solution), for $n = 2$ we get $2!+1! = 2\times1+1 = 3$. For $n = 3$ we get $3!+2!+1! = 3\times2\times1+2\times1+1 = 9$ (solution). For $n = 4$ we get $4! + 3! + 2! + 1! = 33$. For $n = 5$ we get $5! + 4! + 3! + 2! + 1! = 153$. Note that 5! is divisible by 10 (its factors include 5 and 2), and therefore it has the same final digit as the previous value. The same applies for all other higher factorial values (larger than 5) – they will always be divisible by ten, therefore they will end with a zero and they won't change the last digit of the sum. The final digit for $n \geq 5$ will be always 3. Now we use the fact that an integer square never ends in a 3 (the last digits are 1, 4, 9, 6, 5, 6, 9, 4, 1, 0, and this sequence repeats itself). The only solutions are therefore the two we found initially ($n = 1$ and $n = 3$).

Problem 35
There are two points A and B on a line. We draw two circles so that the first circle touches the line in point A, the second touches the line in point B and both circles touch each other in point M. If we were to draw points M for all possible pairs of circles that satisfy the given condition, what shape would they form? A) a line, B) the perimeter of a square, C) the perimeter of a circle, or D) the perimeter of a triangle?



Solution 35
Let's add the tangent common to both circles in the image. This tangent intersects AB in the point S. The distance $|SA|$ is the same $|SM|$ since lines SA and SM are both tangents to the first circle through the same point. Similarly $|SM| = |SB|$. S is therefore the midpoint of AB. Additionally, wherever point M is, the distance $|SM|$ is a half of the distance $|AB|$, and therefore it is constant. once more – the points A and B are given, therefore S is also given. The point M can move, but we know the distance $|SM|$ is constant. It is therefore the radius and all possible points M lie on the circumference of a circle.