MatX 2017 Solutions

matx.p-mat.sk

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Problem 1

Matilda did an experiment in her class. She asked the question: “There is a rectangle with an area of 210 cm^2. Its sides have integer lengths. What is the circumference of this rectangle?”. Matilda got 7 different answers. Is it possible that all of these answers were correct?

Solution 1

We can figure out these eight rectangles fulfil Matilda’s criteria: 1×210, 2×105, 3×70, 5×42, 6×35, 7×30, 10×21, 14×15. Each of these rectangles has a different circumference. Hence it is possible all seven (it could even be 8) answers were correct.

Problem 2

We have the following sequence of numbers:
1, 2, 3, 4, 5,
7, 9, 11, 13, 15,
18, 21, 24, ...
What is the 80-th number in this sequence?

Solution 2

At the end of the first row we have the 5th member of the sequence – number 5. At the end of the second row we have the 10th member of the sequence – number 15. At the end of the third row we have the 15th member of the sequence – number 30. The 80th member of this sequence will be at the end of 80 / 5 = 16th row. And we see that the last number on the n-th row is by $5 \times n$ larger than the last number on the previous row. Therefore the last number on the 16th row is determined by the sum 5 + 10 + 15 + 20 + 25 + 30 + 35 + 40 + 45 + 50 + 55 + 60 + 65 + 70 + 75 + 80 = 680.

Problem 3

We call a multi-digit number “optimistic” if all its digits increase left to right. We call a multi-digit number “pessimistic” if all its digits decrease left to right. There is a pessimistic number P and an optimistic number Q. Numbers P and Q consist of the same digits. The sum of P and Q is 109900 and the difference of P and Q is 84942. What is the value of the pessimistic number P?

Solution 3

Numbers P and Q have to be 5-digit numbers. Even the largest 4-digit numbers are not enough to get a 6-digit sum. If the numbers would be 6-digit, their difference would begin by at least the digit 2.
P and Q are therefore 5-digit and they have the same digits. In other words, Q is just P written backwards. Let Q = abcde. Then P = edcba and a < b < c < d < e. Let’s use P + Q = 109900 = abcde

+ edcba. We see that a + e = 10, because the last digit of the sum is 0. Also from the difference we see that e – a = 8, hence a = 1 and e = 9.
Similarly we know that b + d + 1 = 10 (the +1 is just a carry over) and also b – d + 1 = –4. From this we have b = 2 and d = 7. Now we have P = 97c21. From the sum P + Q we easily conclude, that c = 4 and P = 97421.

Problem 4
A fireman noticed smoke coming from behind a mountain one day. However, he didn't know where the fire was as there are three villages behind the mountain: Truthcester (where people always tell the truth), Liestead (where people always lie) and Alternsby (where people alternate between telling the truth and lying: they tell the truth in the first sentence, lie in the second, truth in the third, and so on). Before the fireman had a chance to do something, the phone rang: "Come quickly, there is a fire here." "Where," asked the fireman. "In Alternsby!" answered an excited voice. To which of these three villages should the fire brigade go?

Solution 4
First we find out who has called. It couldn't be a citizen of Truthcester, because then his second sentence would be a lie. Also it couldn't be a person from Alternsby. The first sentence would imply that the fire is in Alternsby, but the second sentence (which needs to be a lie) contradicts that. Hence the caller needs to be from Liestead.

From the first sentence we know the fire isn't in Liestead. From the second one we know the fire isn't in Alternsby. The fire is therefore in Truthcester.

Problem 5
Our class planned a field trip after finishing the MatX competition. Some students were discussing the length of the trip and they claimed that it is 16, 28, 32, 37 and 15 km long. They were wrong, however, by 5, 7, 8, 9 and 14 km. What was the true length of the trip?

Solution 5
The smallest guess is 15 km, so it needs to be an underestimate. The largest guess is 37 km, so it needs to be an overestimate. The real length of the trip has to be one of the numbers 15 + 5 = 20, 15 + 7 = 22, 15 + 8 = 23, 15 + 9 = 24 a 15 + 14 = 29 (smallest guess plus the deviations). Also the real length needs to be one of the numbers 37 – 5 = 32, 37 – 7 = 30, 37 – 8 = 29, 37 – 9 = 28, 37 – 14 = 23. The only number that occurs in both is 23. We can also check that the remaining deviations from 23 km are correct: 16 + 7 = 23, 28 – 5 = 23, 32 – 9 = 23. The trip was 23 km long.

Problem 6
Wallace wrote a calculation on the blackboard, but he made a mistake: 550 + 461 + 359 + 341 = 2017. Gromit wanted to fix the mistake. He started to look for an unknown number such that if we added this number to each of the five numbers on the blackboard, the calculation would become correct. What was this unknown number?

Solution 6
The sum on the left side is 550 + 461 + 359 + 341 = 1711. If we add a number x to each of these numbers, the sum will increase by 4x. On the right side we have 2017, which we will increase by x. So we solve an equation 1711 + 4x = 2017 + x. From this we have x = 102.

Problem 7

The sides of a cube are labelled using digits from 1 to 6, each digit being used exactly once. There is exactly one digit on each side of the cube. If we look at the cube from three different directions, we always see a different triple of sides. The sums of these triples are 9, 9 and 13. Which number is opposite to number 2 on the cube?

Solution 7

We have figures 1, 2, 3, 4, 5, 6. Only 2 different triples give a sum of 13: 2 + 5 + 6 and 3 + 4 + 6. Also there are only 3 triples giving a sum of 9: 1 + 2 + 6, 1 + 3 + 5 and 2 + 3 + 4 When we write number 2 on the cube, we can try the possibilities using the triples above. By this we can reach 2 different solutions (numbers in a pair occur on the opposite sides of the cube): 1–4, 2–3, 5–6 or 1–2, 3–6, 4–5.

Problem 8

Tess drew a line segment AB and placed a point C on it. Then she drew a point D so that the lengths of the segments AB, CD and BC were all equal and the angles ADC and CBD had the same size. What was the size of the angle BAD?

Solution 8

Sides BC and BD have the same length, so triangle BCD is isosceles. So angles CDB and DBC have the same size, say x. From the sum of the angles in a triangle we have size of BCD = 180 - 2x. Also the size of angle DCA = 2x. We also know the angles CDB and CDA have the same size, so the size of CDA is x. From the sum of angles in triangle CAD we see the size of angle CAD is 180 – 3x. Sides CD and AD have the same length, so CAD is an isosceles triangle – so angles CAD and DCA have the same size. So 180 – 3x = x. From this we see x = 32. As the size of angles BAD, CAD and DCA is 2x, we get that the size of angle BAD is 72 degrees.

Problem 9

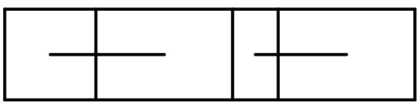
Find a five-digit number x for which the following is true: if we write a digit 1 to the left of the number x we get a number three times smaller than what we would have gotten if we wrote a digit 1 to the right of the number x.

Solution 9

If we write 1 to the left of number x, we get 100000 + x. If we write 1 to the right of the number, we get 10x + 1. So we need to solve an equation 3(x + 100000) = 10x + 1. The solution is x = 42857. We can also check that 3 × 142857 = 425871.

Problem 10

Matthew divided a 42 cm long strip of paper into four different rectangles. He then connected the centres of these rectangles with line segments as shown in the picture. What is the sum of the lengths of these segments in cm?



Solution 10

In each of the 4 rectangles, the segment in the middle has a half of the length of the respective rectangle's side. So the sum of segment lengths needs to be a half of the sum of sides of the rectangles (which add up to a strip 42 cm long). Hence the length of the two segments is 42 / 2 = 21 cm.

Problem 11

There is a two-digit number where the following is true for its digits: if we add their sum and their product, we get the original number as the result. What is the rightmost digit of this number?

Solution 11

The number with this property can be defined as 10a + b, where a is the first digit, and b the second digit of the number. The request indicates that a + b + ab = 10a + b. By simplifying this equation we get ab = 9a. Figure a cannot be zero, as the number 10a + b has two digits, therefore the equation implies that b = 9. The second digit of the number with the given property is the digit 9.

Problem 12

There are 10 white, 8 red and 11 black marbles in a box. What is the smallest number of marbles we need to take out of the box (we don't see what marbles we are taking out of the box) to be certain that there are at least two marbles of each colour among the ones we have taken out?

Solution 12

The largest possible number of marbles when the condition is not yet met is when we have all the marbles of two colours and one marble of the third colour. What is this number in this case? It can be 10 + 8 + 1 = 19, or 10 + 11 + 1 = 22, or 11 + 8 + 1 = 20. We see that largest number is 22. If we had been absolutely unlucky and taken out 10 white, 11 black and one red marble, the condition wouldn't yet be met. However by adding 1 marble it is certainly met and therefore the answer in this case is 23.

Problem 13

Adam and Bob made a trip to the town Root Upon Square. One of them took a bike, the other took a car (the car was faster than the bike). They began the journey at the same time. At one moment the following situation emerged: If Adam had so far travelled half the distance than he actually did, he would now have to travel three times the distance to the destination than he actually has to. Similarly if Bob had so far travelled twice the distance than he actually did, he would now have to travel a three times shorter distance to the destination than he actually has to. Who travelled by bike?

Solution 13

Let's assume that the total length of the trail which Adam and Bob have to travel is 1 km. Let's define the distance that Adam still has to travel at the specific moment mentioned. Let this moment be the present. If Adam had travelled two times less so far, he would have travelled (1 – x) / 2. From the problem we know that he would have to travel three times more than he still has to travel now, so the distance would be 3x. Therefore he would have travelled 1 – 3x thus far. So 1 – 3x= (1 – x) / 2, which implies that x = 1/5. Adam therefore has to travel one-fifth of the total length of the trail.
By similar reasoning we conclude that Bob still has to travel three-fifths of the total length of the trail. That means that Adam travelled more than Bob and therefore it is Bob who is travelling by bike.

Problem 14

When the day after tomorrow becomes yesterday, then today will be as far from Sunday as the day, which was today, when the day before yesterday was tomorrow. What day is it today?

Solution 14

When the day after tomorrow (ATW) becomes yesterday, this will be a day that we can call

after-after-tomorrow (AATW). When the day before yesterday (BY) was tomorrow (TW), it was the day that we can call before-before-yesterday (BBY). According to the problem these days should be equidistant from the Sunday. Looking at the chart BBY-BY-Y-T-TW-ATW-AATW, we see that the days of BBY and AATW are spaced by 6 days. Thus, the only day from which they are equidistant is T. This is Sunday according to the problem which implies that today = T = Sunday.
Note: If you interpreted the problem in a slightly different way, it was possible to get "Tuesday" as the answer. We accepted both Sunday and Tuesday as correct answers.

Problem 15

There is a ten-digit number ABCDEFGHIJ with all its digits distinct. It is also true that the two-digit number AB is divisible by 2, BC is divisible by 3, CD is divisible by 4 and so on up to IJ which is divisible by 10. What is the largest number that satisfies these conditions?

Solution 15

From the pieces of information given we can find out the following digits:

- Since IJ is divisible by ten, J must be 0.
- DE is divisible by five, and since 0 is already occupied, E must be the 5.
- EF, now known as 5F, is divisible by six, so F must be the 4.
- Because of divisibility by 8, 6, 4 and 2 the digits H, F, D and B must be even.
- FG, now known as 4G, has to be divisible by seven. Thus 4G must be either 42 or 49, however all even digits are already taken by B, D, F, H and J, therefore G must be 9.
- GH, now known as 9H, must be divisible by eight, so H must be the 6.
- HI, now known as 6I, must be divisible by nine, so I must be 3.

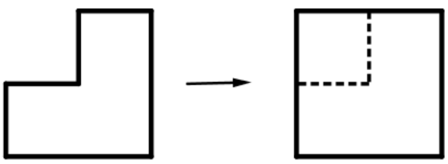
So far, the practice indicates that our number is ABCD549630. As the searched number is to be largest possible, let's try the highest possibility for the digit A. That is 7, since A must be odd digit and 9 is already taken. Similarly, the highest possibility for the digit B is 8. This brings the number 78CD549630. Since D has to be even, there is only one possibility – 7812549630 – and it is easy to verify that this number satisfies all the conditions. The searched number is therefore 7812549630.

Problem 16

A square field with a side of 500 m is divided by straight paths into 625 square gardens of dimensions 20 × 20 meters. Mr Gardner started his walk at a crossroad of two paths and he walked along the paths for 1 km while returning to the place where he had started. What is the maximum number of gardens which are "enclosed" in the circular path Mr Gardner took?

Solution 16

First, it should be noted that the path that encloses the maximum number of paths has to be convex. This means that when we connect any two points within the path with a segment, this segment does not go out of the enclosed area. If the path was non-convex, it could be changed so that more gardens were enclosed, but had the same length (see the figure).



From convex shapes the only appropriate ones are rectangles and squares, as Mr. Gardner is moving along paths only. Let us define the dimensions of the searched rectangle (or square) 20x and 20y. Multiples of 20 are necessary because the path encloses the gardens of dimensions 20 × 20. According to the problem, the path has a length of 1 km, therefore 2 × (20x + 20y) = 1000, that is, x + y = 25. The product xy is to be the largest possible as it determines the number of the enclosed gardens. Trying out all possibilities we get that the largest sum is given by x = 13 and y = 12, which corresponds to 156 gardens.

Problem 17

Among all five-digit numbers which don't contain the digit 0, find the number with the largest difference between this number and the product of its digits.

Solution 17

We know with certainty that the searched number will begin with digits 99. This is because even if we take the smallest number that satisfies the given conditions – 99,111 – the difference of this number and the product of its digits is 99,111 – 81 = 99,030. This difference is greater than 99,000, thus it cannot be overcome by any number that is smaller than 99,000, as the second term of the differences is always positive.

Now let's try to continue in this way. Other such number is 99911, which has the difference 99,911 – 729 = 99,182. The difference increased again. Adding another nine (99,991) however, has an adverse effect, because the product of the digits of this number is 6,561, which reduces the difference below 99,000, which is obviously worse than what we already have. In the last two places we want to leave the digit 1. We already know that the first two digits must be nines. Now we only need to determine the middle digit. What would happen if we replaced it with a smaller digit k? The number would be reduced by (9 – k) × 100, whereas the product of the digits would be reduced by (k – 9) × 81. We see that the difference would be reduced, so we want to keep the 9 in third place. Number 99,911 is therefore the solution of this problem.

Problem 18

In the word PIKOMAT replace each of the individual letters with a different digit from 1 to 9 so that the resulting seven-digit number is divisible by 72 and is the smallest such number.

Solution 18

For a number to be divisible by 72, it must be divisible by nine and eight. The lowest number that we can try is 1234567. This is not divisible by eight, so we need to increase the number. Although the next number, 1234568 is divisible by eight, it is not divisible by nine. Sequential adding of 1 will eventually bring us the number 1234584, which is divisible by 72. Even though this number is a multiple of 72, the digit 4 is repeated, thus this number does not satisfy the conditions. However, since we already have a multiple of 72, we can keep adding 72 and verifying whether the resulting number does not contain repetitive digits. The first number that satisfies all conditions is 1237896.

Problem 19

The house where Hugo and Albus live is long – it has multiple entrances. In each entrance there are 4 apartments on each floor. All apartments in this house are numbered by consecutive natural numbers. The numbering starts on the 1st floor of the first entrance and ends on the top floor of the last entrance. That is, apartments on the first floor of the first entrance have numbers 1 to 4, apartments on the second floor of the first entrance have numbers 5 to 8 and so on. There are shops on the ground floor and these are not numbered. Hugo lives on the fifth floor in apartment no. 83 and Albus lives in a different entrance on the third floor in apartment no. 169. How many floors does the house have?

Solution 19

Let's define the number of floors in the house p. The fact that Hugo lives on the fifth floor in apartment number 83 indicates that the apartments 81–84 are on the fifth floor, apartments 77–80 on the fourth floor, apartments 73–76 on the third floor, apartments 69–72 on the second and apartments 65–68 on the first floor. This means that the numbering in the previous entrance ends with 64. Therefore 64 must be divisible by 4 × p, since 4 × p is the number of apartments in a single entrance.

Albus lives on the third floor in apartment 169. Apartments 169–172 are thus on the third floor, apartments 165–168 on the second floor and 161–164 apartments on the first floor of the entrance. The numbering of flats previous entrance thus ends with the number 160, therefore 160 must be divisible by 4 × p.

So we have two conditions of divisibility: p must divide without a remainder the numbers 16 and 40. 16 = 2 × 2 × 2 × 2 and 40 = 5 × 2 × 2 × 2, so p can only be 2, 4 or 8. Given that Hugo lives on the fifth floor, we see that p must be greater than or equal to 5, therefore we have only one option: p = 8. The house has therefore 8 floors.

Problem 20

As Andrew walked down the street he was counting his steps. He got a five-digit number which didn't have any digit 0 or 1 in it. Andrew told us that the number of even digits in this number is even. The second, the third and the fourth digits are smaller than 4 and there are exactly two pairs of consecutive digits that are both the same. The fourth digit is the number of twos in the number. What was the number of steps Andrew counted?

Solution 20

Let's define the five-digit number we are searching for as ABCDE. Since there is even number of even digits and the total number of digits is five, there can only be zero, two or four even digits in the number. As there is the digit 6 in the number, the number of even digits is either two or four.

The fourth digit (D) indicates the number twos in number. As the 2 is even and the number of even digits (including six) is two or four, number of twos must be 1 or 3. However, since the number doesn't have the digit 1, figure D can only be equal to 3. There are therefore three twos in the number. If any of those was to be on the last place in the number (that is, if E = 2) and the condition of adjacent pairs of identical digits was to be applicable, number ABCDE would have to be 22,332 and there would be no digit 6 in it. This case can not occur, so E cannot equal to 2.

Now we have only three positions where we can place twos – in places A, B and C. The searched number thus becomes 2223E and must contain the digit 6 so we get that ABCDE = 22,236. It is easily verified that this number meets all conditions, which means that we have found the only possible solution.

Problem 21

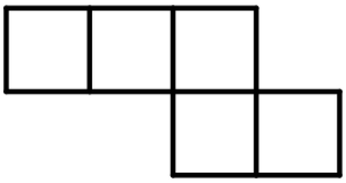
John learned to count on his fingers in the following manner: He counted only on one of his hands one by one. He started on the thumb, continued with the index finger, the middle finger, the ring finger and he got the number 5 upon reaching the pinkie. Then he continued with the ring finger (6), the middle finger (7), the index finger (8), the thumb (9), back to the index finger (10), the middle finger (11), and so on. One day he wanted to count up to 2017. On which finger did he reach 2017?

Solution 21

The following sequence of eight terms repeats itself in John's counting: the thumb (1), the index finger (2), the middle finger (3), the ring finger (4), the pinkie (5), the ring finger (6), the middle finger (7), the index finger (8). The remainder on dividing 2017 by 8 is 1, hence the number 2017 will correspond to the first term in the sequence – the thumb.

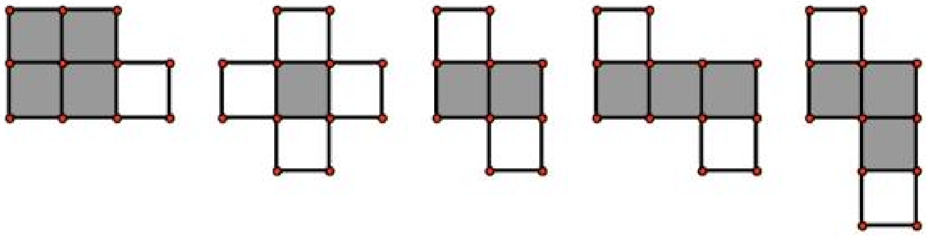
Problem 22

John has a puzzle. Each piece of the puzzle is made of five squares which are connected together by one or more sides. You can see one of the puzzle pieces in the image below. How many different pieces of such puzzle are there? We don't consider rotated or flipped pieces as different.



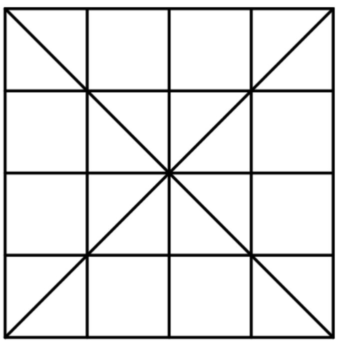
Solution 22

Let us start by defining two terms: denote by a "trunk" a square that is adjacent to other squares by at least two sides and denote by a "leaf" a square that is adjacent to other squares by one side exactly. See the picture below for all possible orderings of trunks. For each of these cases, we can easily find the number of puzzle pieces with this trunk ordering. In the same sequence as in the picture below, there is 1, 1, 3, 4 and 3 of them (think about it!). As a result, there are exactly 12 different puzzle pieces.



Problem 23

How many triangles are there in the picture below?



Solution 23

To make sure that we will count every triangle, we need a good counting system. Without loss of generality, assume that the side length of each of the small squares is 1 (thus the dimensions of the large square are 4×4). All of the triangles in the picture have a right angle. Possible side lengths of the triangles are 1, 2, 3, 4 and $\sqrt{2}$, $2\sqrt{2}$, $3\sqrt{2}$ a $4\sqrt{2}$. Possible pairs of the leg lengths of the triangles are (1, 1), (2, 2), (3, 3), (4, 4), ($\sqrt{2}$, $\sqrt{2}$), ($2\sqrt{2}$, $2\sqrt{2}$). The corresponding areas to these pairs of leg lengths are 0.5, 2, 4.5, 8 and 4.

Now that we know all of the possible dimensions and areas of the triangles in the picture, let us count them: there are 16 triangles of area 0.5, 4 triangles of area 1, 12 triangles of area 2, 4 triangles of area 4, 8 triangles of area 4.5 and 4 triangles of area 8. That is, there are 16 + 4 + 12 + 4 + 8 + 4 = 48 triangles altogether.

Problem 24

How many numbers from 1 to 10000 have at least one digit 5 in them?

Solution 24

Because the number 10000 doesn't contain the digit 1, it is enough to consider all numbers between 1 and 9999. First, consider only numbers between 1 and 99. Among each ten numbers starting with the same digit (except for 50–59) there is exactly one number that contains a five. Among 50–59, all ten numbers contain a five. Therefore there are 9 × 1 + 10 = 19 numbers that contain a five among 1–99. Now consider only numbers between 1 and 999. Among each hundred numbers starting with the same digit (except for 500–599) there is exactly 19 numbers that contain a five. Among 500–599, all hundred numbers contain a five. Therefore there are 9 × 19 + 100 = 271 numbers that contain a five among 1–999. Last, consider all numbers between 1 and 9999. Among each thousand numbers starting with the same digit (except for 5000–5999) there is exactly 271 numbers that contain five. Among 5000–5999, all thousand numbers contain a five. Therefore there are 9 × 271 + 1000 = 3439 numbers among 1–9999 that contain a five.

Problem 25

One mile has 5280 feet. The rails on a railroad are 30 feet long. The speed of the train in miles per hour is an integer. Whenever the train passes over the point where the rails are joined, there is an audible click. A railroad worker Peter is on the train and is counting the clicks. How long (in terms of seconds) must he count the number of clicks so that the number of clicks is the same as the speed of the train in miles per hour? Note: when the train passes over n rails, Peter counts this as hearing n clicks.

Solution 25

The length of a rail is 30 feet, therefore there are 5280 / 30 = 176 rails in a mile. Let the speed of the train in miles per hour be n. The train will pass over 176n rails in an hour. Put differently, the train will pass over 176n rails in 3600 seconds, i.e. n rails in 3600 / 176 seconds. Peter must therefore listen to the clicks for 3600 / 176 = 20.45 seconds.

Problem 26

A red and a blue six-sided dice were rolled. Out of all possible outcomes, in how many cases would the product of the two rolled numbers be greater than 10?

Solution 26

6 × 6 = 36 possible pairs could have been rolled. The simplest approach is to list all of these and calculate the product for each option – the product is greater than 10 in 17 of the cases.

Problem 27

In the square ABCD the length of AB is 10 cm. Let S be the centre of the square. Let X be a point on AB so that AX = 7 cm. Let Y be a point on BC so that the angle YSX is right. What is the area of the quadrilateral XBYS?

Solution 27

Let K and L be the centres of the sides AB and BC respectively. This gives rise to two triangles: KXS and LYS. We know the area of the triangle KXS because |KX| = 2 cm and |KS| = 5 cm. Denoting the angle BXS by α, we see that the angle BYS equals 90 – α since the two remaining angles in the quadrilateral BYSX are right and the sum of angles in a quadrilateral is 360. This implies that the size of YSL is α because YLS is right and the sum of angles in a triangle in 180. Similarly, we know that the angle KSX has a size of α because SKX is a right angle and the sum of angles in a triangle is 180. This in turn means that the triangles KXS and SYL are identical, and hence have the same area. The area of the quadrilateral XBYS hence equals to the area of KBLS – area of KXS + area of SYL = area of KBLC = 25 cm^2.

Problem 28

They sell delicious chocolate for 10 pounds on the Island of Sodor. A coupon is enclosed with each chocolate. One can buy another chocolate for three such coupons. What is the smallest number of pounds we need to buy 500 chocolates?

Solution 28

First, let us show that 3340 pounds is enough. We can buy 334 chocolates with these, getting 334 coupons on the purchase. Let us keep 1 of the coupons aside and use the remaining 333 coupons to buy 111 chocolates. This gains 111 new coupons which we use to buy 111 / 3 = 37 chocolates. Getting 37 new coupons, let us put 1 of them aside and use the remaining 36 coupons to buy 12 chocolates. We use the 12 new coupons to buy 4 chocolates, receiving 4 new coupons. We now have 4 + 2 coupons (the 2 coupons are the ones which we had put aside) which we can use to buy 2 chocolates. We will receive two new coupons and have no use for these, as by now we will have bought 334 + 111 + 37 + 12 + 4 + 2 = 500 chocolates.

We now only need to show that we have to spend at least 3340 pounds (and no less) to buy 500 chocolates. What is the value of chocolates in pounds? We know that 3 Coup = 1 Choc + 1 Coup (as we get 1 chocolate and 1 coupon for 3 coupons), i.e. 2 Coup = 1 Choc. Furthermore, 10 Pounds = 1 Choc + 1 Coup (as we get 1 chocolate and 1 coupon for 10 pounds). Substituting 1 Choc, we get 10 Pounds = 3 Coup, that is 1 Coup = 10 / 3 Pounds. Therefore, one coupon is worth 10/3 pounds and one chocolate is worth 20 / 3 pounds. 500 chocolates are worth 500 × 20 / 3 = 3333.33 pounds as a result. It is clear that we have to pay at least as much as the value of the chocolates in pounds. Because we are always paying in multiples of 10 pounds, the closest value we can pay is 3340 pounds.

Problem 29

Find the smallest natural number which ends in 56, it is divisible by 56 and the sum of its digits is 56.

Solution 29

It follows from the divisibility by 56 that the number we are looking for must be divisible by 7 and 8. To make the number divisible by 8, its last three digits have to be divisible by 8. This means that the last three digits of the number we are looking for must be 256, 456, 656 or 856. The sum of the number's digits has to be equal to 56. Because we are looking for the smallest number with these properties, we need it to have as few digits as possible. We can achieve this by using large digits. The last triple of digits will therefore be 856. The digits' sum is 19, we therefore need the sum of the remaining digits to be 56 – 19 = 37. The shortest set of digits that satisfies this is 1, 9, 9, 9, 9. The number 1999856 is not divisible by 7 and hence it is not divisible by 56. The sum of digits is 56, so to get a number with the required properties, we can decrease one of the digits and increase another one by the same amount. The last triple has to stay the same and we can't increase the nines any further, so we have to increase the digit 1. Let us try to increase the 1 to 2. By trial and error, we find out that by decreasing the second nine by one, we get a number divisible by 7, i.e. 29899856 is the solution.

Problem 30

Find the largest natural number where each pair of consecutive digits forms a two-digit prime number while all these prime numbers are distinct. Number with such properties is for instance 11731, since 11, 17, 73 and 31 are all distinct prime numbers.

Solution 30

The largest such number is 619737131179 which we can find out as follows: Firstly, we find that all primes starting with 2, 4, 5, 6 or 8 can't be placed anywhere after the first digit because then the previous pair of consecutive digits couldn't form a prime. The number (apart from the first two digits) therefore has to be composed of the following primes: 11, 13, 17, 19, 31, 37, 71, 73, 79, 97. So the number can have at most 11 + 1 = 12 digits (we can use a prime starting with one of the forbidden digits at the start of the number). The only task left is to find a sequence of the primes above so that we use all of them. This can be done by trial and error where we can reduce the number of options by some further reasoning (for example if we notice how many times each digits occurs as the first and as the last digits of the individual primes).

Problem 31

There is a number 2017. Write one digit to the left and one digit to the right of this number so that the final number is as large as possible and divisible by 68.

Solution 31

A number is divisible by 68 if it is divisible by 17 and by 4. We don't have a rule for divisibility by 17, but we know that a number is divisible by 4 if its last two digits are divisible by 4. Because 2017 ends with a 7, the only digits we can write to the right of 2017 are 2 and 6 to make the result divisible by 4.
Let us now consider the first digit. We want our number to be as large as possible, so let us try 9 for the first digit. We are hence trying whether 920176 or 920172 is divisible by 17. Luckily, 920176 is indeed divisible by 17 and therefore by 68 and hence satisfies the conditions.

Problem 32

What is the number of zeros the product 1 × 4 × 7 × ... × 94 × 97 × 100 ends with?

Solution 32

We get a zero at the end of a product only if we multiply by 10 or by 5 and 2. So we only need to count how many fives and twos are included in the prime factorization of the product we are looking at and the smaller of these two amounts will give us the number of zeros (for example, if the product only contained 3 fives and 2 twos, it would end in two zeros).
There will surely be more twos in the product than fives because every other number in the product is even, so we have at least 16 twos. So let us look at the number of fives.
Fives are included in the numbers divisible by 5; the first such number is 10. Because the numbers in the product always increase by 3, the next number divisible by 5 will be 10 + a multiple of both 5 and 3 which is 15. Hence the numbers divisible by 5 are: 10, 25, 40, 55, 70, 85, 100.
Let us count how many fives they include: 10 (1), 25 (2), 40 (1), 55 (1), 70 (1), 85 (1), 100 (2), that is 9 fives in total. There are no more fives in the product and we have at least 9 twos in the product, so the product will end in 9 zeros.

Problem 33

After getting straight A's for her A-Levels, Mishka spends the summer doing a summer job at the Cold Beach swimming pool. She sells numbered tickets to the visitors. The number depends on the order in which the visitors came: The first visitor of the day gets a ticket with number 1, the second visitor with number 2, the third with number 3, and so on. At some point during the day, Mishka ran out of yellow paper which the tickets were printed on. So she had to use red paper from that moment on. By the end of the day, she sold the same number of yellow tickets as the red tickets. She noticed in the evening that the sum of the numbers on the yellow tickets was 1681 smaller than the sum of the numbers on the red tickets. How many tickets in total did she sell on that day?

Solution 33

Let us look at both of the sums (yellow and red): The yellow sum is the sum of numbers from 1 to a number x. The red sum runs from x + 1 to 2x. Let us write these on top of each other:

Yellow: $1 + 2 + 3 + 4 + ... + (x - 2) + (x - 1) + x = Y$

Red: $(x + 1) + (x + 2) + (x + 3) + (x + 4) + ... + (x + x - 2) + (x + x - 1) + (x + x) = R$

From the above, we can see that the red sum is the same as the yellow sum with each of the terms increased by x. Therefore, we know that $R = Y + x \cdot x$. We also know that $x \cdot x = 1681$, hence $x = 41$ which means that Mishka sold 82 tickets in total.

Problem 34

Jess, Tess and Bess participated in a competition. There were 12 participants in total. After the competition was over, the girls were bored so they were counting the number of possibilities in which the competition could have ended. Jess was confident and she thought that she might end up among the first five participants. On the other hand, Tess thought she hadn't done very well and therefore she thought that she will be among the four last ones. Bess had been eavesdropping on the organisers of the competition but she wasn't entirely sure what she heard. She was certain though, that she will be either 2nd or 12th. How many possible orderings of the participants are there if we take Jess', Tess' and Bess' thoughts into the account?

Solution 34

Let us split the problem into two cases: 1) Bess came second, 2) Bess came twelfth.

In the first case, Jess would come 1st, 3rd, 4th or 5th and Tess would come 9th, 10th, 11th or 12th. There are therefore $4 \times 4 = 16$ possibilities in which the match could have ended.

In the second case, Jess would come 1st, 2nd, 3rd or 4th and Tess would come 9th, 10th or 11th. There are therefore $5 \times 3 = 15$ possibilities in which the match could have ended.

In total, there are 15 + 16 possible results for Tess, Jess and Bess. However, we must also consider the remaining participants. They could have taken the remaining 9 places in any possible order, i.e. in any of the $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ possible orders. There are therefore $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 31 = 11249280$ possible results in total.

Problem 35

In a room, there are six chairs in two rows with three chairs in each row. In how many ways can we seat two Cambridge students, two Oxford students and two Harvard students on the chairs so that we don't have the embarrassing situation of a Cambridge student sitting next to an Oxford student?

Solution 35

Let us first assign each chair to a school so that no Oxford student needs to sit next to a Cambridge student. After we have done this, there will be 2 ways in which each school can sit its students on the two chairs, so there will be $2 \times 2 \times 2 = 8$ ways to seat the students in total.

Now let us see how we can assign the chairs to schools. There are two cases: 1) Both Cambridge students will sit in the same row 2) Each of the Cambridge students will be in a different row.
In case 1), there are two ways of choosing which row the Cambridge students will sit in. Then there are three ways of assigning two chairs in this row for Cambridge students. Hence $2 \times 3 = 6$ options in total. After this, the Oxford students will take seats in the row where there are no Cambridge students – this is possible in 3 ways. So there are $2 \times 3 \times 3 = 18$ options in total. The Harvard students just take the remaining chairs.
In case 2), no Cambridge student can take the middle seat, because there would be no way of seating the Oxford students. Hence there are two ways of choosing a chair for a Cambridge student in each row, i.e. $2 \times 2 = 4$ options in total. In this case, the Harvard students will take the middle seats and the Oxford students will take the remaining edge seats.
All in all, there are $(2 \times 3 \times 3 + 2 \times 2) \times 2 \times 2 \times 2 = (18 + 4) \times 8 = 22 \times 8 = 176$ ways of seating the students.

Problem 36

There are 1000 marbles in a row. One of them is white, all remaining are black. We start removing some marbles. We will do it this way: First, we remove all marbles that are on the odd positions. There will be 500 marbles left. Then we again remove marbles which are now on the odd positions. We will repeat this until there is only one marble left. What position should we place the white marble on if we want it to be the last marble to stay on the table?

Solution 36

In the first round, we remove all the marbles in odd positions. In the second round, if we use the old numbering, we remove the following marbles: 2, 6, 10, 14, ..., 994, 998, i.e. all marbles in positions divisible by 2 but not divisible by 4. In the next round, we remove marbles no. 4, 12, 20, ..., 988, 996, i.e. marbles in positions divisible by 4 but not divisible by 8.
In each round, we double the position number of the first marble to remove and then keep adding twice this number and removing marbles in the resulting positions. The position numbers of the first marble to remove in each round are: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512 (and nothing further because the next number would be 1024 and there are only 1000 marbles).
This means that in the last round, we take marble no. 512. The next marble to remove would be marble no. $512 + 1024 = 1536$, but there is no such marble. Hence this is the last round and no. 512 is the last marble removed. Therefore 512 is where we want to position the white marble.

Problem 37

ABCD is a trapezoid with bases AB and CD and P is the intersection of its diagonals. The area of the triangle ABP is 25 cm^2 and the area of the triangle PCD is 16 cm^2. What is the area of the entire trapezoid in cm^2?

Solution 37

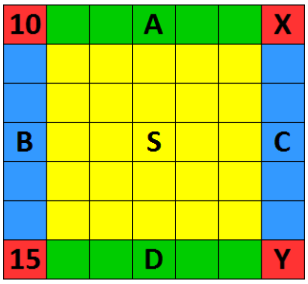
Let $|AB| = a$, $|CD| = b$, let the height of AB in the triangle ABP be v_a and the height of the side CD in CDP be v_b. Using alternating angles, we get that the angles BAC and ACD are equal and the angles ABC and BDC are equal. As a result, the triangles ABP and CDP are similar (their internal angles are the same but they are of different dimensions). Because of this, the ratio of the sides of ABP and CDP is the same as the ratio of their heights. Therefore $v_a / v_b = a / b$, so we have $av_b = bv_a$. We also know that the ratio of the areas gives $25 / 16 = av_a / bv_b$. Substituting the first into the second relationship gives $25 / 16 = v_a v_a / v_b v_b$, hence $5 / 4 = v_a / v_b$, hence $v_b = 4/5\ v_a$. The area of the trapezoid we are trying to find is $(v_a + v_b)(a + b) / 2$ or multiplying out: $av_a / 2 + bv_b / 2 + av_b / 2 + bv_a / 2$. The values of the first two terms in the sum are known because they are the areas of the triangles ABP and CDP. Using $a v_b = bv_a$ we can reduce the remaining two terms into one: av_b. Because $v_b = 4/5\ v_a$, we get $av_b = 4/5\ av_a = 8/5\ av_a / 2 = 8/5 \times 25 = 40$. The total area of the trapezoid is therefore $16 + 25 + 40 = 81\ cm^2$.

Problem 38

There is a square grid 7 × 7 with a number in every cell. The sum of any four cells that share a common vertex is the same in the entire table. There is a number 10 in the upper-left corner cell and a number 15 in the lower-left corner cell. What is the difference of the numbers written in the upper-right corner cell and the lower-right corner cell?

Solution 38

Let us split the grid into 8 sectors as in the picture. We are interested in the difference X – Y.



Look at the top left corner and sectors A, B, S and then at the bottom left corner and sectors B, D, S. We know that their sums have to be equal because they both contain 9 quadruples of the constant sum. So we have: 10 + A + B + S = 15 + B + D + S. This is the same as 10 + A = 15 + D, or A – D = 5. For the right hand part of the table, we similarly get X + A + C + S = Y + C + D + S, i.e. X + A = Y + D, or A - D = Y – X. Because we know that A – D = 5, we have Y – X = 5.

Problem 39

Shaun and Bitzer sold their flock of sheep. They sold all of them at the same time and they got as many euros for each sheep as was the total number of sheep in the flock. They got the money in ten-euro notes and in coins. After they divided the ten-euro notes using the system "one goes to me, one goes to you", they were left with one ten-euro note. Shaun suggested: "I will take the remaining ten-euro note and you can have all the coins." Bitzer didn't agree with that: "No way, I would get less money than you." So Shaun made the offer better: "Ok, I will also write you a cheque for such an amount that we will be even." Bitzer agreed to that. What was the amount on the cheque in euros?

Solution 39

We know the following about the total amount of euros received for the sale: 1) The amount is a perfect square 2) The amount is composed of an odd number of tens and a remainder which is smaller than 10. Let us know focus on the perfect squares with an odd digit in the tens place: 16, 36, 256, ... Seeing this list suggests that if a perfect square has a 6 in the ones place, the digit in the tens place is odd. We can easily prove this by listing the last two digits of perfect squares:

01, 04, 09, 16, 25, 36, 49, 64, 81, 00,
21, 44, 69, 96, 25, 56, 89, 24, 61, 00,
41, 84, 29, 76, 25, 76, 29, 84, 41, 00,
81, 64, 49, 36, 25, 16, 09, 04, 01, 00,
01, 04, 09, 16, 25, ... etc.

As visible from the table, the sequence repeats from the fifth row on. This means that we have covered every option in the table above and our statement above was correct. As a result, we know that the amount of money left over after splitting the ten euro notes was 6 euros. Bitzer therefore got 4 euros less than Shaun (6 euros instead of 10). The cheque had to settle the difference of 2 euros (Bitzer lost two euros and Shaun got two extra euros which made them even).

Problem 40

How many three-digit numbers are there such that all of their digits are distinct and the digit in the tens place is smaller than the digits in the hundreds and ones places?

Solution 40

Let us consider the options by the digit in the tens place. There can't be a digit 9 or a digit 8 in the tens place – the largest digit that can be there is 7. For 7 in the tens place, there are two possibilities: 978 and 879.

If there is a digit 6 in the tens place, there are more options: we can choose from three options in the hundreds place and from two options in the ones place (we can't use the same digit twice). For a digit 6 in the tens place, we therefore have $3 \times 2 = 6$ possibilities.

For the digit 5 in the tens place, it will be similar: $4 \times 3 = 12$ possibilities, etc. up the digit 1 in the tens place where there will be 9×8 possibilities. In total, there will be $2 + 6 + 12 + 20 + 30 + 42 + 56 + 72 = 240$ possibilities.

Problem 41

Mr Bolt got the newest model of the flying carpet. When flying x kilometres per hour, the flying carpet needs only x/10 litres of high-quality Chinese tea per 100 km as a fuel. So for instance, when flying 28 km/h it needs 2.8 litres per 100 km. What is the maximum distance from the castle Mr Bolt can go to, if he needs to be back in one hour and he has only 1 litre of high-quality Chinese tea available? We assume Mr Bolt flies with constant speed all the time.

Solution 41

Let us start by expressing the consumption of the carpet better. If the carpet uses up x / 10 litres per 100 km then it uses up x litres per 1000 km.

Suppose Mr Bolt decided to fly 10 km away from the castle. His speed would have to be 20 km/h for him to be back in an hour (he has to fly 10 km there and 10 km back). The carpet's consumption would be 20 litres per 1000 km. Because he would only make a distance of 20 km, he would use up 0.4 litres of tea. This is a good attempt, but there is still 0.6 litres of tea left, so there must be a better solution.

Let us try the same method in general: If Mr Bolt decided to fly x km away from the castle, his speed would need to be 2x km/h. The carpet's consumption would be $x \cdot 2x / 1000$, i.e. $2x^2 / 1000$ litres. We now Mr Bolt had 1 litre of tea, so $2x^2 / 1000 = 1$. This gives $x = \sqrt{1000}/2 = 15.81$ km.

Problem 42

Rob drew a few points in the plane. Each of these points is either yellow, blue or red. There is at least one point of each colour. No three points lie on a straight line. Can we always find a triangle such that it has one yellow vertex, one blue vertex and one red vertex and such that the triangle doesn't have any other of Rob's points inside it?

Solution 42

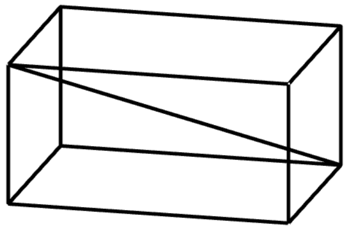
Choose any three points of different colours. These three points form a triangle. Now, two things can happen:

1) There are no points inside the triangle. If this is the case, the condition from the problem is met.

2) There is a point P of colour B inside the triangle. Take the point P and make a new triangle from the two points of the original triangle whose colours are different to B and from the point P. We now have a new, smaller triangle which still has vertices of all three colours. Case 2) will repeat itself until case 1) becomes true. This has to happen, because the number of points is finite.

All in all, this shows that the condition from the problem will always hold.

Problem 43
A cuboid 30 × 40 × 50 is cut into 60 cubes of size 10 × 10 × 10. How many of these cubes does the cuboid's space diagonal cross?



Solution 43
The space diagonal needs to cross three depth layers of the cube, four width layers of the cube and five length layers of the cube. Because the layers have the same dimension in each direction, these crossings will be regular in each direction. I.e. the diagonal will cross the depth layers in 1/3 and 2/3 of the diagonal's depth. Similarly, the diagonal will cross the width layers in 1/4, 2/4 and 3/4 of the diagonal's width and the diagonal will cross the length layers in 1/5, 2/5, 3/5 and 4/5 of the diagonal's length. Because none of these fractions are equal, each crossing will just cross a layer in one direction. Also, each crossing of a layer means crossing a new cube. There are 2 + 3 + 4 = 9 crossing, but we also have to include the starting cube. The space diagonal will hence cross 10 cubes in total.

Problem 44
A letter was delivered to a company saying that they should send trucks to the train station to get the requested delivery of 10 tons of supplies. The supplies are packed in boxes, each box not being heavier than 1 ton. The company owns trucks which can carry at most 3 tons. What is the smallest number of trucks that should be sent to the train station so that it is certain that they will be able to bring back the entire delivery in one go?

Solution 44
At the first sight, it looks like the answer to this problem should just be 4 because the trucks have a total weight capacity of 12 tons. However, it's not necessarily true that each truck uses its full weight capacity. For example, suppose each box weighted 10/13 tons. Each of the trucks could only carry 3 of these boxes (because 4 boxes of 10/13 tons weight approximately 3.08 tons together) so five trucks would be necessary.
Are five trucks always enough? We know that each truck can carry at least 2 tons. If a truck's load is less than 2 tons, we can add another box to it because the box weights less than 1 ton. Now we have 5 trucks carrying 2 tons each, hence together, then can carry at least 10 tons. Therefore five trucks are always enough.

Problem 45
What is the smallest number of lines that can divide a plane into 1000 sectors?

Solution 45
This problem was very difficult and aimed to get you thinking if you got this far!

If there is only a single line in the plane, it splits the plane into two sectors. If we add another line (not parallel to to the original line), the plane will be split into 4 sectors. What happens if we add another line, not parallel to either of the two previous lines, and intersecting the two previous lines in exactly two points? The new line will be split into three parts by the two points of intersection. Each of these parts will split an already existing sector of the plane into two. Hence the new number of sectors will be 7.

Now suppose there are already n mutually non-parallel lines in the plane so that no point is an intersection of more than two lines. If we add a new line which is not parallel to any o the existing lines and which doesn't cross any of the existing intersections, the new line will be split into n + 1 parts. Each of the parts will split an existing sector of the plane into two. I.e. there will be n + 1 new parts of the plane. At the same time, this is the highest possible number of sectors that can be created by adding a single line because the new line cannot cross more than n sectors of the plane.

If there is one line in the plane, the plane is split into 2 sectors. Two lines can split the plane into at most 4 sectors. Three lines into 2 + 2 + 3 = 7 sectors, four lines into 2 + 2 + 3 + 4 = 11 sectors, five lines into 2 + 2 + 3 + 4 + 5 = 16 sectors, and so on. Calculating this series until 45, i.e. until 2 + 2 + 3 + ... + 45, we get 1036 which is the first time when the sum goes over 1000. Therefore the minimum number of lines required to split the plane into 1000 sectors is 45.