MatX 2018 Solutions

matx.p-mat.sk

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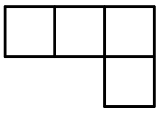
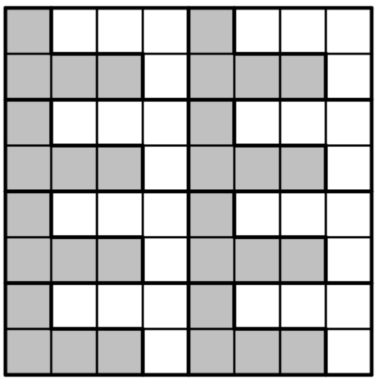
Problem 1

A number is written in every cell of a table 8 × 8 so that the sum of numbers in every four cells which are in the shape of "L" (see the picture) is 20. What is the sum of all the numbers in the table? Note: You can rotate or flip the "L" shape.



Solution 1

If we draw the table we quickly find out that we can fully cover it without overlaps with the L-shapes as shown in the picture. The sum of numbers in each L-shape is 20 and we used 16 L-shapes to cover the entire table. Therefore the sum of all numbers in the table has to be 16 × 20 = 320.



Problem 2

Ella wrote down in her diary that drawing 5 cars takes her the same time as drawing 10 houses, drawing 8 houses takes the same time as 5 squares, drawing 10 squares takes the same time as 12 circles and drawing 6 circles takes the same time as 16 hearts. If she has enough time to draw 7 cars, how many hearts would she be able to draw in that amount of time?

Solution 2

Let's rewrite the exercise into equations so that it is easier to think about it:

(1) 5C = 10H

(2) 10H = 5S

(3) 10S = 12C
(4) 6C = 16♡
Rearrange (4): 12C = 32E. We then know that 10S = 32♡.
Rearrange (2): 16H = 10S. We then know that 16H = 32♡.
Rearrange (1): 8C = 16H. We then know that 8C = 32♡, which can be simplified to 1C = 4♡. Therefore 7C = 28♡. Ella would manage to draw 28 hearts.

Problem 3

Jacob fell into a well so he started to climb up. He climbed up 1/18 of the well's height every day. However, he fell down 1/36 of the well's height every night when he was taking a rest. How many days did it take Jacob to climb out of the well?

Solution 3

Well, well, well. Jacob climbs up 1/18 – 1/36 = 1/36 of the height of the well every day. However, he gets out of the well in 35 days as he doesn't fall the 1/36 on the 35th day, since he is already out of the well by the night.

Problem 4

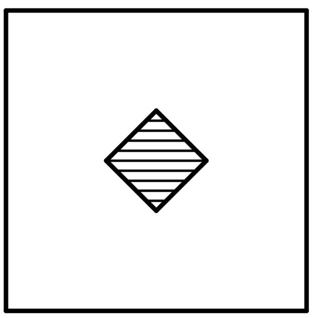
50 teams participate in a sports tournament. They are divided into 5 groups with 10 teams in each group. Every team plays a match against every other team in the same group. Once these have been played, the winning team in every group goes to the second round of the competition. In the second round, the winning team from each group plays a match against every other winning team. How many matches will be played altogether?

Solution 4

If there are X teams in a group, then they will play X × (X – 1) / 2 against each other. This is because we have X possibilities how to choose the first team and (X – 1) possibilities how to choose the second team. However, in order not to count A vs B and B vs A as two distinct matches, we need to further divide the result by two. There were 10 teams in every of the 5 groups. In each group every team played every other team in the group, hence there had to be 10 × 9 / 2 = 45 matches, therefore together 5 × 45 = 225 matches. Then the winning teams had to play 5 × 4 / 2 = 10 matches, therefore all together 225 + 10 = 235 matches.

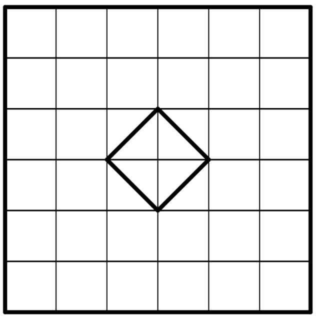
Problem 5

John's sister spent a lot of time before Christmas cutting snowflakes out of paper. John wanted to join in, so he got a square sheet with sides of length 6 cm. He wanted to be original, so instead of cutting out triangles or half circles he just cut out one small square. He cut it out from the middle of the big square so that the centre of the small square was at the same place as the centre of the sheet. The small square was rotated relative to the big square by 45 degrees and its diagonal was 2 cm long (as the picture). What is the area of the snowflake that John made in cm^2?



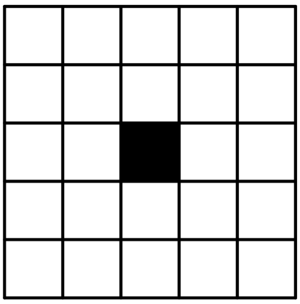
Solution 5

We can draw the big and the small square on a square grid 6 × 6 like in the image. Thanks to the grid it is obvious that the area of the small square is 2 × 2 / 2 = 2 cm^2 and the volume of the big square is 6 × 6 = 36 cm^2. Now we just need to compute the difference between the area of the big square and the small one which we cut out, getting 36 – 2 = 34 cm^2.



Problem 6

Samuel has a 5 × 5 grid like the one in the picture. The middle cell is black. How many squares which contain the black square inside them are there in the grid?



Solution 6

Let's do this systematically. The squares can be of sizes 1 × 1, 2 × 2, 3 × 3, 4 × 4 a 5 × 5. We can easily compute how many squares satisfy the given criteria for each size:

1 × 1: 1
2 × 2: 4
3 × 3: 9
4 × 4: 4
5 × 5: 1
Together 1 + 4 + 9 + 4 + 1 = 19.

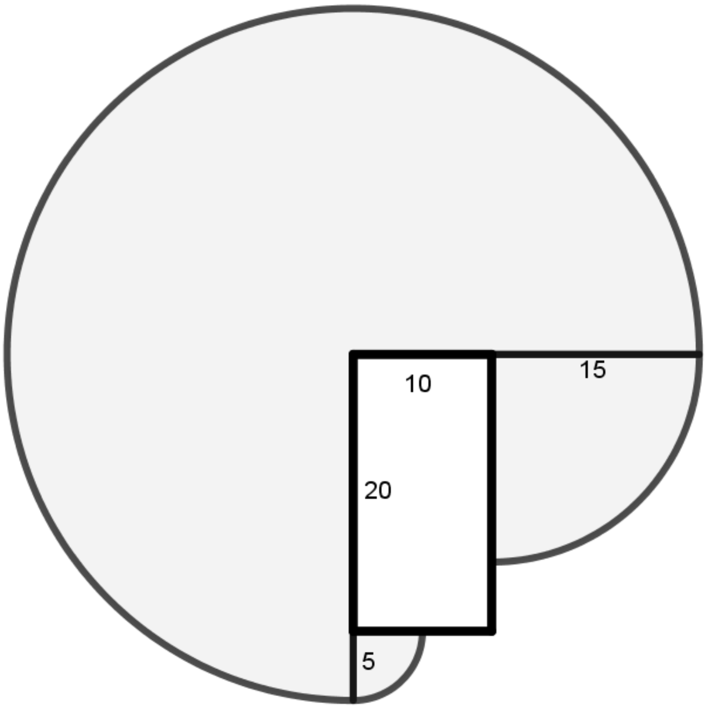
Problem 7

A horse is tied outside at the corner of a rectangular stable. The size of the stable is 10 by 20 meters. What is the area in m^2 the horse can walk to if it is tied on a 25 m rope? Submit the answer rounded to one decimal place. Note: The table below lists areas of circles with various radii:

5 m: 78,5 m^2
10 m: 314,2 m^2
15 m: 706,9 m^2
20 m: 1256,6 m^2
25 m: 1963,5 m^2

Solution 7

We see clearly from the picture that the area the horse can walk on is 3/4 of a circle with radius of 25 m and 1/4 of circles with radii of 15 m and 5 m. Therefore: 3/4 × 1963,5 + 1/4 × (78,5 + 706,9), which is approximately 1669 m^2.



Problem 8

Three brothers arrived at a hotel. They were tired so they immediately fell asleep. Meanwhile, a cook brought them a pot of potatoes. The first brother woke up and was hungry, so he ate one third of the potatoes and went back to sleep. Then the second brother woke up and he didn't know his first brother has already eaten, so he also ate one third of the potatoes in the pot. Then the third brother woke up and since he thought he heard something, he assumed one of his brothers has already eaten, so he ate one half of the potatoes in the pot. There were 6 potatoes left in the pot. How many potatoes were in the pot at the beginning?

Solution 8

Let there be X potatoes at the beginning. The first brother ate X/3, hence the amount of potatoes left after his meal was X – X/3 = 2X/3. The second brother ate one third of the rest, so 1/3 × 2X/3 = 2X/9. The amount of potatoes left after his meal was then 2X/3 – 2X/9 = 4X/9. The third brother ate one half of the rest, so 1/2 × 4X/9 = 2X/9. The amount of potatoes left after his meal was then 4X/9 – 2X/9 = 2X/9 potatoes. And it is not hard any more. We know there were 6 potatoes left, so 2X/9 = 6, so the amount of potatoes at the beginning was X = 27.

Problem 9

We have three mutually distinct rectangles of area 12 cm^2 which have the following dimensions expressed in cm as whole numbers: 1 cm × 12 cm, 2 cm × 6 cm, 3 cm × 4 cm. Find out the smallest area in cm^2 where we would get four mutually distinct rectangles as a solution in a similar exercise.

Solution 9

The simplest solution for this exercise is to simply go one by one over all numbers greater than 12 until we find a solution. We can simplify our work by skipping all prime numbers (which have only 2 divisors while we need at least 7 of them). After a bit of searching we get to the number 24 which satisfies the given condition since 24 = 1 × 24 = 2 × 12 = 3 × 8 = 4 × 6.

Problem 10

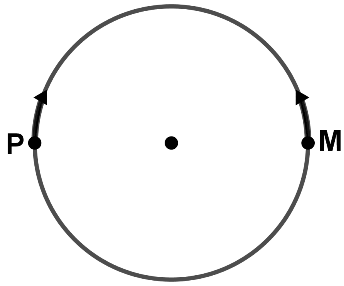
Our neighbour Rosie has four cats of different colours. She got the cats at different times from her friends. Children would like to know what are the cats' names and which one she got the third. However, the neighbour has a bad memory, so she was only able to tell them this:
"I got the cat Diana first.
I got the black cat before the grey cat and I got Albina, the while cat, as the second.
I got Cilka before I got the brown cat.
Bella is not grey."
Which cat did the neighbour get as the third one and what colour is it?

Solution 10

We know straight from the exercise that:
1 – Diana – ?
2 – Albina – white
3 – ? – ?
4 – ? – ?
We know immediately that Cilka had to be the third one, as otherwise it could not be before the brown one. We also know that the fourth one had to be brown:
1 – Diana – ?
2 – Albina – white
3 – Cilka – ?
4 – ? – brown
The black one had to be before the grey one, so the first one had to be black and the third one had to be grey. The last name we have left is Bella, so it logically had to be the fourth one:
1 – Diana – black
2 – Albina – white
3 – Cilka – grey
4 – Bella – brown

Problem 11

Pat and Mat are running on a circular running track. Both of them run in opposite directions. They start in such places that the line connecting them forms the diameter of the circular track. Since the moment they started to the moment they first met, Pat ran 100 m. Since the moment they first met to the moment they met for the second time, Mat ran 150 m. How long is the circular running track in meters? Note: The speeds of Pat and Mat are constant.



Solution 11

When Pat and Mat met for the first time they had together run one half of the length of the track. When they met for the second time they had together run the full length of the track which necessarily had to take them twice as long as to run the half of the length of the track. When running the whole length Mat did 150 m. So if he ran for half the time he would have done 75 m. Therefore the length of the half of the track is 100 m (which Pat ran) + 75 m (which Mat ran) = 175 m. The length of the whole track is therefore 175 + 175 = 350 m.

Problem 12

There was a party where every woman danced with exactly two men during the party and every man danced with exactly three women during the party. There were 18 men at the party. How many women were at the party?

Solution 12

There were W women and M = 18 men at the party. Let's count the number of dances that happened at the party. Every woman danced with two men, therefore there must have been 2 × Z dances. But we also know that every man danced with 3 women, therefore there must have been 3 × M dances. Therefore 2 × Z = 3 × M. M = 18, so 2 × Z = 54, so there must have been Z = 27 women at the party.

Problem 13

When asked about his birthday, Thomas said: "The day before yesterday I was only 25 and next year I will turn 28." This is true only one day in a year. When is Thomas's birthday?

Solution 13

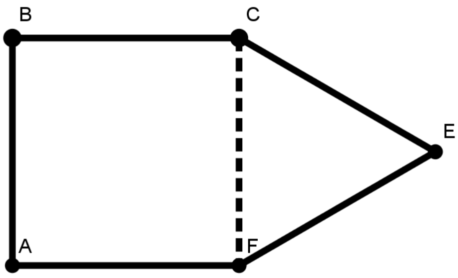
The solution is 31. 12. (2017) and Thomas is telling us the fact on 1. 1. (2018). He was 25 the day before yesterday (30. 12. 2017), he was already 26 on 31. 12. (2017). This year he will be 27 on 31. 12. (2018) and next year he will be 28 on 31. 12. (2019).

Problem 14

Andrew has a picture of a pentagon ABCDE which has all sides of equal length. Moreover, the angles ABC and BAE have 90 degrees and all other angles inside the pentagon ABCDE are smaller than 180 degrees. What is the size of the angle AED?

Solution 14

If we draw all we know from the problem itself we get the following picture. Clearly, ABCE is a square. Since all sides of the pentagon are of the same length, then CDE has to be an equilateral triangle, with all its internal having 60 degrees. Therefore the size of the angle |AED| = 90 + 60 = 150 degrees.

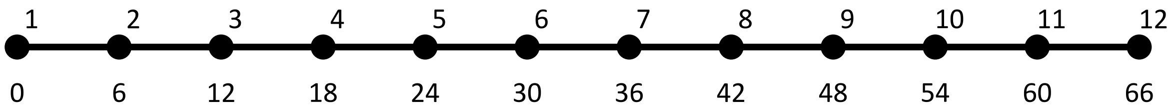


Problem 15

The clock struck six times. It took 30 seconds from the first stroke to the last one. How many seconds is it going to take if the clock is to strike 12 times?

Solution 15

The best way to solve this problem is by drawing a picture. We can clearly see from it that 12 strokes will take 66 seconds. The reason why it is not 60 seconds as one would expect is that one stroke doesn't take 6 seconds, but the time between 2 strokes is 6 seconds.



Problem 16

We roll two dice, one is blue and the other is red. What is the probability that the sum of the rolled numbers will be 9?

Solution 16

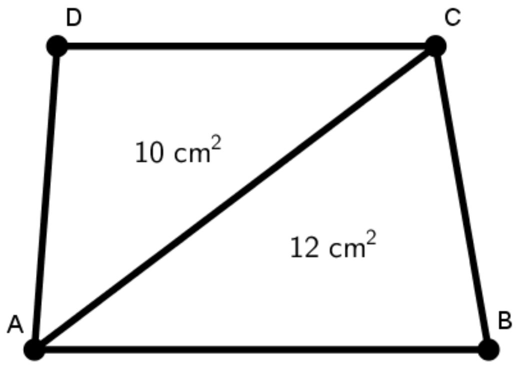
If we roll two dice we have 36 possible rolls because the dice have different colours (rolling 1 on the red die and 6 on the blue die is not the same as rolling 6 on the red die and 1 on the blue die). The following combinations out of these 36 combinations give the desired sum of 9: 3 + 6, 6 + 3, 4 + 5, 5 + 4. That is 4 combinations. The probability is then 4/36 = 1/9.

Problem 17

A trapezoid is divided by its diagonal into two triangles with areas 10 cm^2 and 12 cm^2. Its longer base is 6 cm long. What is the length of the shorter base of this trapezoid in centimetres?

Solution 17

The area of the upper triangle is the length of the shorter base times the height of the trapezoid divided by two. The area of the lower triangle is the length of the longer base times the height of the trapezoid divided by two. That implies that the area of the lower triangle has to be larger and therefore 12 cm^2. That immediately implies that the height of the trapezoid has to be 4 cm (since 4 × 6 / 2 = 12 cm^2), and then the length of the shorter base has to be 5 cm (since 4 × 5 / 2 = 10 cm^2).



Problem 18

Alice went for a walk in carrot fields. As she walked, she wandered: “How many fields could there be around me?” She got the following pieces of information at the local information centre: the number of carrot fields is the smallest 6-digit number ABCDEF divisible by five so that only A and D are prime numbers and its digits are in decreasing order (A > B > C > D > E > F). How many carrot fields were there? Note: 1 is not a prime number.

Solution 18

The number ABCDEF has to be divisible by five. Therefore the last digit has to be either 0 or 5. However, 5 can’t be the last digit since A would then have to be equal to 10. So we know that F = 0. A and D are prime numbers. So A has to be one of the numbers 2, 3, 5, 7. It can’t be 2, 3 or 5 since then there would be no numbers left for B, C, D, E. Therefore A = 7. Now let’s look at D: It can’t be 5, as there would not be enough numbers left for B and C. It has to be 2 or 3 then. We are searching for the smallest solution, so let’s try D = 2 first. So far, we’ve got 7BC2E0. E can be only 1 (which is not a prime number), B and C can be only 6 and 4. So the solution is 764210. Just to be sure, let’s verify that for D = 3 we won’t get a better solution. 7BC3E0. E = 1, because 2 is a prime number. B = 6, C = 4. Although this also satisfies the given conditions, the first solution is smaller. Therefore the solution is really 764210.

Problem 19

We call a natural number N beautiful if N is a prime number, N – 14 is a prime number and also N + 14 is a prime number. Find the sum of all beautiful numbers.

Solution 19

A prime number a number which is divisible only by 1 and itself. So if we show that some number is divisible not only by these two numbers but by also at least one other divisor, then we immediately know that this number is not a prime number. It doesn’t make sense to use divisibility by two in this problem as all prime numbers except two are odd. So how about using divisibility by three? We want to therefore show that N – 14 and N and N + 14 are all not divisible by three. Let’s assume N – 14 gives remainder 1 when divided by three. But then N has to be divisible by three and hence it is not a prime number. So let’s assume N – 14 gives remainder 2 when divided by three. But the N + 14 has to be divisible by three. So the only way a solution can exist is if N – 14 is divisible by 3 and it is a prime number at the same time. The only case when this holds is when N – 14 = 3. Now we just need to verify that this is a valid solution: N = 17, N + 14 = 31 which are both primes, therefore N = 17 is the only solution of this problem.

Problem 20

We call a three digit number scary if it is divisible by six and if we remove any one of its digits we get a two digit number also divisible by six. How many scary three digit numbers are there? Note: We don't consider 00, 01, 02, ..., 09 as two digit numbers.

Solution 20

Let the digits of the 3-digit scary number be a, b, c. Thanks to the rules of divisibility we know that ab, ac, bc and abc have to be all divisible by 2 and 3 at the same time. We also know that the rule for divisibility by three says that a number is divisible by three if and only if the sum of its digits is divisible by three. We therefore know that the following have to be all divisible by 6: 1) a + b + c, 2) a + b, 3) a + c, 4) b + c. This implies that a, b, c have to be all divisible by three. The reason is that if we choose any two numbers, their sum has to be divisible by three. But when we add the third number to them the sum has to still be divisible by three. Therefore the third number has to be also divisible by three. And since this has to hold for all three numbers, all three of them have to be divisible by three.
Now let's look at the digit c. It has to be divisible by three, but also by two, otherwise the whole number abc won't be divisible by six. The digit c can be therefore 0 or 6. The digit b can be only 6 since it can't be 0. The digit a can be 3, 6 or 9, because in neither of numbers ab, ac, bc it is not the final digit. The solution is therefore these numbers: 360, 366, 660, 666, 960, 966.

Problem 21

We have an arithmetic machine that does the following:

1. It gets a number x as its input.
2. It multiples x by two.
3. It adds two to the result from the previous step.
4. It multiples the result from the previous step by itself and returns this result.

Joseph placed his favourite number in the machine and he got out a number which was the same as the number that Matthew got out of the machine when he placed his favourite number in it. Does it necessarily mean that their favourite numbers are the same?

Solution 21

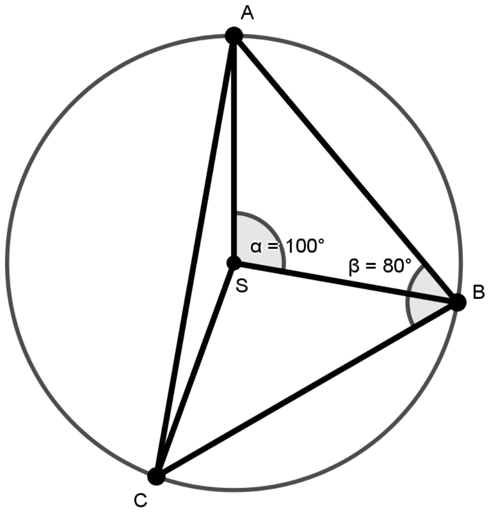
Let's rewrite what the arithmetic machine does in mathematical form. In the first step it turns x into 2x. In the second step it turns 2x into 2x + 2. Then it turns 2x + 2 into (2x + 2) × (2x + 2). Now it is important to remember that 2 × 2 = 4, but also –2 × –2 = 4. So for instance for x = 1 we get (2 × 1 + 2) × (2 × 1 + 2) = 16 and for x = –3 we get (2 × –3 + 2) × (2 × –3 + 2) = 16. The answer is therefore "no", as we showed that one can put two distinct numbers into the machine while getting the same result back.

Problem 22

Stella has three friends: Alex, Bob and Cyprian. By coincidence, the straight routes from Stella to every one of them have the same length. Stella found out that the routes to Alex and Bob form an angle of exactly 100 degrees. Bob found out that his routes to the other two boys form an angle of 80 degrees. Cyprian was sad that nobody thought about his routes. Find out what is the angle between Cyprian's routes to Alex and Bob.

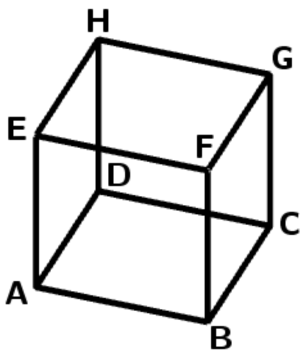
Solution 22

We know that |ASB| = 100°. Then we know that |SA| = |SB| = |SC|, aand that |ABC| = 80°. Since the triangle ASB is isosceles, the angles SAB and SBA have to be of equal size (180° – 100°)/2 = 40°. That implies that also the size of the angle SBC will be 40° and hence |BSC| = 100°. Since |SB| = |SC|, the triangle SBC will be isosceles, hence the angles |SCB| = |SBC| = 40°. It has to be the case that |ASC| + |ASB| + |BSC| = 360°, and |ASB| + |BSC| = 200°, therefore |ASC| = 160°. Since the triangle SAC is also isosceles then the angle |SCA| = (180° – 160°)/2 = 10°. Now we just need to add |SCA| + |SCB| = |ACB| = 10° + 40° = 50°, which is the size of the angle of the routes from Cyprian to Alex and Bob.



Problem 23

The vertices of the cube ABCDEFGH in the picture are marked with numbers 1 to 8 with the vertex A having the value 1. Suggest such a way to label the vertices of this cube with the numbers so that the sum of the vertices on each side of the cube is the same as all the other sums. Enter the solution as eight numbers in the order ABCDEFGH (A = 1, so the first number will be always 1).



Solution 23

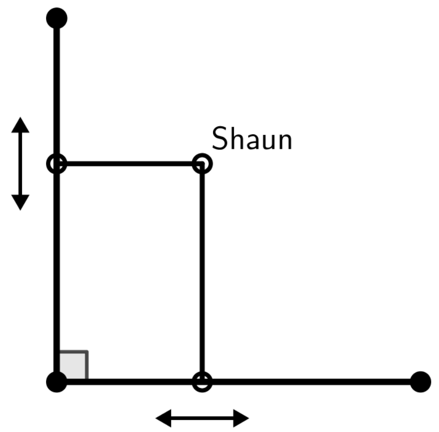
Let's begin by calculating what will be the sum of the numbers on each side of the cube. Since we count each number at each vertex in three sides of the cube, we need to multiply the numbers 1 to 8 by three and then divide by six, which gives us the sum of the numbers on each of the six sides of the cube: (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8) × 3 / 6 = 18. Now we just need to keep trying until we find a combination that satisfies the given condition. There are quite a few solutions, here are all of them (for the case where A = 1):

A B C D E F G H
1 4 5 8 6 7 2 3
1 4 5 8 7 6 3 2
1 4 6 7 8 5 3 2
1 4 7 6 8 5 2 3
1 6 3 8 4 7 2 5
1 6 3 8 7 4 5 2

1 6 4 7 8 3 5 2
1 6 7 4 8 3 2 5
1 7 2 8 4 6 3 5
1 7 2 8 6 4 5 3
1 7 4 6 8 2 5 3
1 7 6 4 8 2 3 5
1 8 2 7 4 5 3 6
1 8 2 7 6 3 5 4
1 8 3 6 4 5 2 7
1 8 3 6 7 2 5 4
1 8 5 4 6 3 2 7
1 8 5 4 7 2 3 6

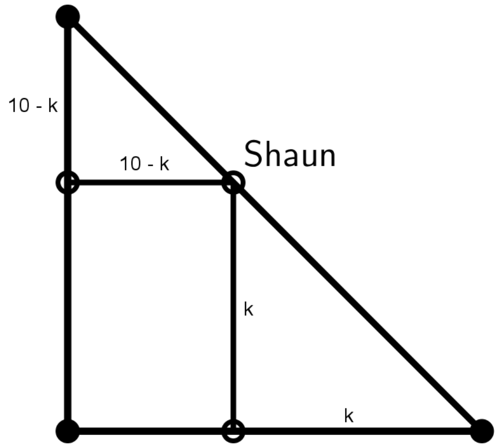
Problem 24

Shaun the Sheep is grazing in a meadow. He is on a rope which is fixed in the following way: there are three poles stuck in the vertices of a right isosceles triangle with its legs being 10 m long. Two of them are connected by two ropes, each 10 m long. Another rope of the same length has a ring on both of its ends and it is connected onto the two ropes using these rings. The sheep is connected with a ring to that rope as shown in the picture. The ropes can move freely through the rings. What is the area in m^2 Shaun the Sheep can graze on? Note: Shaun can't jump over any of the two ropes which connect the poles.



Solution 24

Shaun is able to get at most to the hypotenuse of the triangle, because when he goes k meters away from one leg of the triangle, he has to be at most $10 - k$ meters from the other leg of the triangle which forms two isosceles right triangles with legs of lengths k and $10 - k$ like in the picture. The area of this triangle is $10 \times 10 / 2 = 50$ m^2.



Problem 25

There are three times more children than adults on an airplane. The average age of adults is 44 years. What is the average age of children if the average age of all people on the airplane is 20 years?

Solution 25

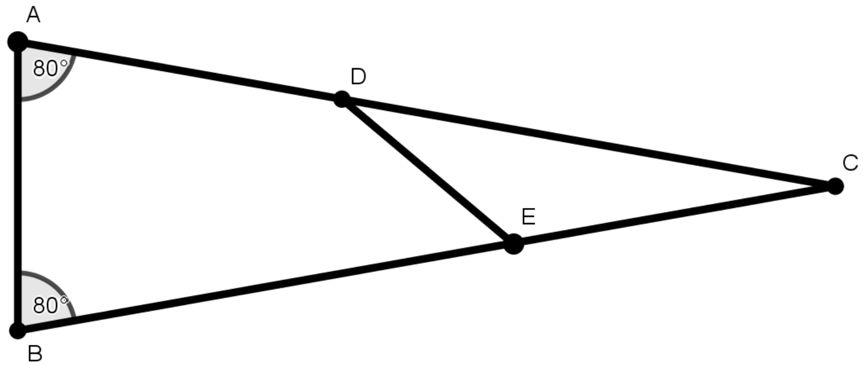
The exercise says that the average age of the adults is 44 years. This has to hold for any any case, therefore also the case where every adult on the plane is 44. The same is true for the children: we can assume that all children are of the same age. How do we calculate an average: We sum a bunch of numbers and we divide the sum by their amount. So let A be the number of adults and 3A the number of children as there is three times more children than there is adults on the plane. Let the average age of the children be X. Then the average of all ages will be (because we assume all adults and all children are of the same age): $(3A \times X + 44 \times A) / (3A + A) = (3X + 44) / 4 = 20$. We can then easily compute that $X = 12$.

Problem 26

In a triangle ABC there is a point E on the side BC and a point D on the side AC so that |BE| = |DC| and |AD| = |EC|. What is the size of the angle DEB if we know that |CBA| = 80° and |CDE| = 30°?

Solution 26

The fact that |BE| = |DC| and |AD| = |EC| implies that |BC| = |AC|, therefore the triangle ABC is isosceles. That means that the angles |CBA| = |BAC| = 80°. The fact that |CDE| = 30° implies that |EDA| = 180° – 30° = 150°. Now let's have a look at the quadrilateral BEDA. We know the sizes of three of its angles (80°, 80°, 150°) and we are supposed to find out the size of the fourth one. The sum of angles in a quadrilateral is always 360°, hence we can easily compute the size of DEB: 360° – 80° – 80° – 150° = 50°.



Problem 27
Joseph loves ice-cream and he regularly visits the local ice-cream parlour. They offer 10 ice-cream flavours, out of which 5 are fruit flavours: banana, lemon, apple, raspberry and red currant; and 5 are classic flavours: vanilla, chocolate, coconut, pistachio and stracciatella. However, Joseph is very picky and when he selects the flavours of his perfect ice-cream he follows the following rules:
1) He never combines fruit and classic scoops. The only exception is the raspberry flavour, which he combines even with the classic flavours.
2) If he's having a fruit flavoured ice-cream, he goes for 3 scoops.
3) If he's having a classic flavoured ice-cream, he goes only for 2 scoops (if he is combining it with raspberry flavour, he goes for one scoop of raspberry and one scoop of the classic flavour).
4) He never goes for more than 2 scoops of one flavour.
5) He never combines scoops of banana with scoops of lemon.
6) He goes for a scoop or scoops of apple only if the combination contains at least on scoop of red currant.
How many different ice-cream combinations can Joseph create? Note: Combination, which is created just by changing the order of the scoops, is considered to be the same.

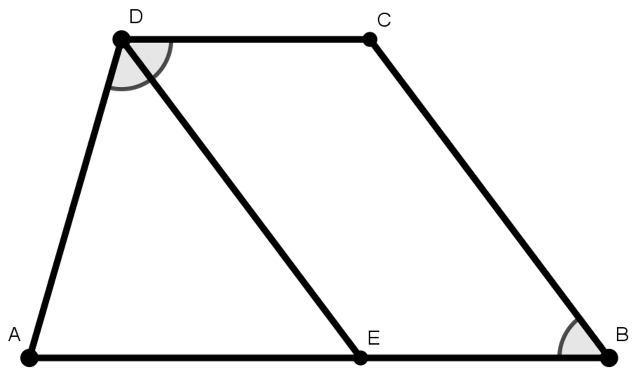
Solution 27
This exercise requires a great deal of patience and attention. Let's divide the ice-creams into three groups:
1) Pure classic flavours.
2) Raspberry + one scoop of a classic flavour.
3) Pure fruit flavours.
We have 15 possible combinations in the first group. In the second one we have only 5. The third group is a bit more involved, so let's write all of them down (note that we use R for raspberry and C for red currant):
BBR, BBC, BAC, BRC
LLR, LLC, LAC, LRC
AAC, ARC
RRB, RRL, RRC
CCB, CCL, CCA, CCR
This gives 4 + 4 + 2 + 3 + 4 = 17 combinations.
Together we then have 15 + 5 + 17 = 37 combinations.

Problem 28
We say a number is happy if it is divisible by 2, 3, 5 or 7 without any remainder. How many numbers between 1 and 100 are not happy?

Solution 28
We are searching for number from 1 to 100 which are not divisible by 2, 3, 5 or 7. It is important to note that all of these numbers are prime numbers. Every number can be written as a product of prime numbers. So is there any number from 1 to 100 that can be written as a product of at least two prime numbers which are not equal to 2, 3, 5 or 7? The smallest such number is $11 \times 11 = 121$ which is larger than 100. Therefore there is no such a number. Now we immediately know that the only numbers that are not happy are prime numbers and the number 1: 1: 1, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97. This is 22 numbers in total.

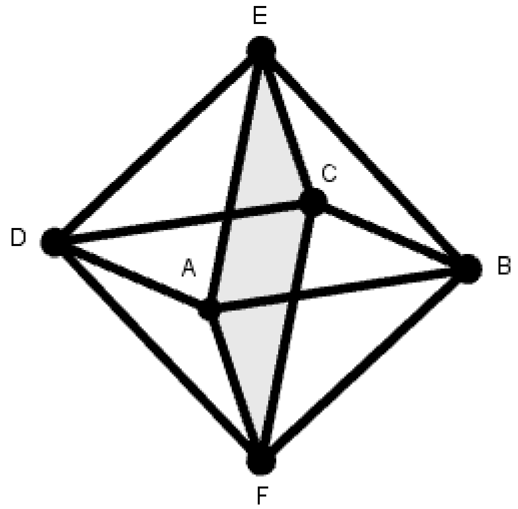
Problem 29
The size of the angle ADS is twice the size of the angle ABC in a trapezoid ABCD with bases AB and CD. The length of the side AD is 4 cm and the length of the side CD is 3 cm. What is the length of the side AB in centimetres?

Solution 29
Let's draw an extra line which is parallel with DB and goes through the point D as show in the image. We will call E the point where this line intersects AB. $|EB| = |DC| = 3$ cm because EBDC is a rhomboid. We also know that $|EDC| = |ABC|$ and by combining this what we know from the exercise we get that $|ADE| = |EDC| = |ABC|$. It is also the case that $|ADE| = |AED|$ because ED is parallel with CB. That implies that the triangle AED is isosceles with legs $|AD| = |AE| = 4$ cm. Now we just add $|AB| = |AE| + |EB| = 4 + 3 = 7$ cm.



Problem 30
ABCDEF is a regular octahedron with its 3 cm long edges. The octahedron is formed by two tetrahedral pyramids ABCDE and ABCDF. What is the area of the quadrilateral EAFC in cm^2?

Solution 30
ABC is a right angle. That implies that also AEC and AFC have to be right, because $|AE| = |EC| = |AF| = |FC| = |AB| = |BC| = 3$ cm and all three triangles ABC, AEC and AFC share the same hypotenuse AC. That immediately implies that the quadrilateral EAFC is a square. Its area is therefore simply $3 \times 3 = 9$ cm^2.



Problem 31

We need to divide 120 Euro between five workers so that the second worker gets as many more Euro than the first worker, as the third worker gets more than the second worker, the fourth than the third and the fifth than the fourth. The first two workers will get seven times less money than the remaining three workers together. How many Euro should the second worker get?

Solution 31

Let a, b, c, d, e be the workers' salaries. We know that b = a + x, c = b + x, d = c + x, e = d + x, where x is the amount by which the next worker gets larger salary than the previous one. Using this we can now express the salary of each worker only by using a and x: b = a + x, c = a + 2x, d = a + 3x, e = a + 4x. We also know that a + b + c + d + e = 120, and that 7 × (a + b) = c + d + e. Now we just need to express b, c, d, e in the two latter equations using a and x: 5a + 10 x = 120 and 7 × (2a + x) = 3a + 9x, therefore 11a = 2x. Plugging this into the previous equation we get 5a + 55x = 120, so 60a = 120, so a = 2 and x = 11. Therefore the first worker gets 2 Euro, the second 13 Euro, the third 24, the fourth 35 and the fifth 46.

Problem 32

Barbara wrote down two whole non-zero numbers. She then added them, subtracted them, multiplied them and divided them. That way she got four results. When she added up those four results she got -1. When she left out the result of addition and added up the remaining three results (i.e. subtraction, multiplication and division results), she also got -1. Which two numbers could have Barbara written down? Enter the numbers sorted in increasing order.

Solution 32

Let's name the two given numbers A and B. Now these two equations hold:

$(A + B) + (A \times B) + (A/B) + (A - B) = -1$

$(A \times B) + (A/B) + (A - B) = -1$

This clearly implies that $A + B = 0$, hence $A = -B$. So instead of A and B we can simply say that the numbers we are looking for are A and $-A$. So let's rewrite the first equation just using A:

$(A + -A) + (A \times -A) + (A / -A) + (A - -A) = -1$

which we simplify to

$-(A \times A) - 1 + 2A = -1$

which we then further simplify to

$-(A \times A) + 2A = 0$

Factoring out A we get:
$A \times (2 - A) = 0$
The solution is then either $A = 0$ or $A = 2$. But $A = 0$ can't be a valid solution since the exercise says that A must be non-zero. The solution is therefore 2 and –2.

Problem 33
Anna and Mary are playing a dice game. Anna rolls a few dice and she wins if she rolls a six on at least one of them. Mary wins if Anna doesn't roll a six on any of the dice. What is the smallest number of dice Anna needs to roll so that she has better chance of winning than Mary?

Solution 33
Let's start with a simple case. What if the girls had only one die. In that case Anna would have a chance of 1/6 while Mary would have a chance of 5/6. So one die is clearly not enough. What if they had two dice? Then Mary needs one not to roll on neither of the dice, so only one of the five numbers 2, 3, 4, 5, 6 is good for her. So her chance is 5 × 5 / 6 × 6 = 25/36 which is greater than 1/2. What is they had three dice? Then Mary would have a chance of 5 × 5 × 5 / 6 × 6 × 6 = 125/216, that she wins which is still larger than 1/2. If they had 4 dice then the chance for Mary becomes 5 × 5 × 5 × 5 / 6 × 6 × 6 × 6 = 625/1296 which is less than 1/2. Anna therefore needs to play with at least four dice.

Problem 34
A man and a woman are in a hurry to catch a plane so they are both running on a moving walkway (in the direction of its travel). The man and the woman have steps of the same length – exactly one meter. The man runs twice as fast as the woman (i.e. he does twice the amount of steps in the same time) and he had to do exactly 28 steps to run across the moving walkway while the woman had to do only exactly 21 steps. How long is the moving walkway in meters?

Solution 34
Solution 1: Let the speed of the moving walkway be s and let the woman's speed be v. Let the length of the moving walkway be l meters. Then the man's speed is $2v$. When the man and the woman moved along the moving walkway, a part of the movement forward was done by the walkway and a part of it was done by them. The ratio between how much they got forward thanks to the walkway and how much thanks to themselves is the same as the ration between their speed and the speed of the walkway. For the man this is: $(l - 28)/28 = s/2v$. Similarly, for the woman this is: $(l - 21)/21 = s/v$. If we solve for s in both equations we get: $s = 2v(l - 28)/28$ and $s = v(l - 21)/21$. Putting both equations together we get $l - 3 \times 28 = 2l - 2 \times 21$ and we can simplify this to $l = 42$.

Solution 2: Imagine that the mowing walkway is longer. If the woman does 21 steps, the man does $21 \times 2 = 42$ steps. The man needs 28 steps to walk the entire length of the mowing walkway, so he walked $42 - 28 = 14$ extra steps which also means he walked extra one half of the length of the moving walkway. The man and the woman are now $42 - 21 = 21$ steps apart, the man walked 1.5 of the length of the moving walkway, the woman 1 length. Therefore 21 steps is the same as the length of one half of the moving walkway. The moving walkway is therefore $2 \times 21 = 42$ steps/metres long.

Problem 35
Mark drew a trapezoid ABCD with bases AB and CD. He marked the intersection of its diagonals P. He found out that the area of the triangle ABP was 16 times larger than the area of the triangle CDP. Then he drew a line parallel with AD through the point C which intersected AB in point E. Similarly, he drew a line parallel with BC through the point D which intersected AB in point F. Compute the ratio between |AE|:|EF|:|FB|.

Solution 35

We can draw the trapezoid and it will look similar to the one in the picture. We know that |AE| = |DC| = |FB|. Triangles ABP and CDP are similar. So the ratio between the height of the triangle ABP and its base AB is the same as the height of the triangle CDP and its base CD. Since the area of a triangle is its base times its height divided by two and the ratio between the triangles' areas is 1:16, the ratio between the lengths of their bases has to be 1:4, i.e. |AB| = 4|CD|. Now we know that |AB| = |AE| + |EF| + |FB| = 4|CD| = |CD| + |EF| + |CD|, so |EF| = 2|CD|. The ratio |AE|:|EF|:|FB| is therefore 1:2:1.